

Reachability in Petri Nets with Inhibitor arcs

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Abstract

We define 2 operators on relations over natural numbers such that they generalize the operators '+' and '*' and show that the membership and emptiness problem of relations constructed from finite relations with these operators and \cup is decidable. This generalizes Presburger arithmetics and allows to decide the reachability problem for those Petri nets where inhibitor arcs occur only in some restricted way. Especially the reachability problem is decidable for Petri nets with only one inhibitor arc, which solves an open problem in [KLM89]. Furthermore we describe the corresponding automaton having a decidable emptiness problem.

1 Introduction

The decidability of the reachability problem in Petri nets without inhibitor arcs is proved in [May84] and later in [Kos84] and [Lam92]. On the other hand, the reachability problem is undecidable for Petri nets with two inhibitor arcs which follows from [Min71]. An open problem in [KLM89] was the reachability problem for Petri nets with one inhibitor arc.

An important method is the use of semilinear sets which are defined using the operators $+$, $*$, \cup over finite sets of vectors (multisets). Semilinear sets are the solutions of Presburger formula, where Presburger arithmetic is the first order logic over the natural numbers and the addition. Presburger arithmetic is decidable and semilinear sets are closed under \cap and complement [GS65],[ES69].

But a reachability relation for a Petri net is in general not semilinear. For that reason the basic idea of this paper is to replace $+$ and $*$ by suitable operations \circ_Q and $*_Q$ which are able to express a reachability relation as the sequence of relations (like the transitive closure used in [Imm87] to characterize NL with first order logic or more generally in [Avr03]).

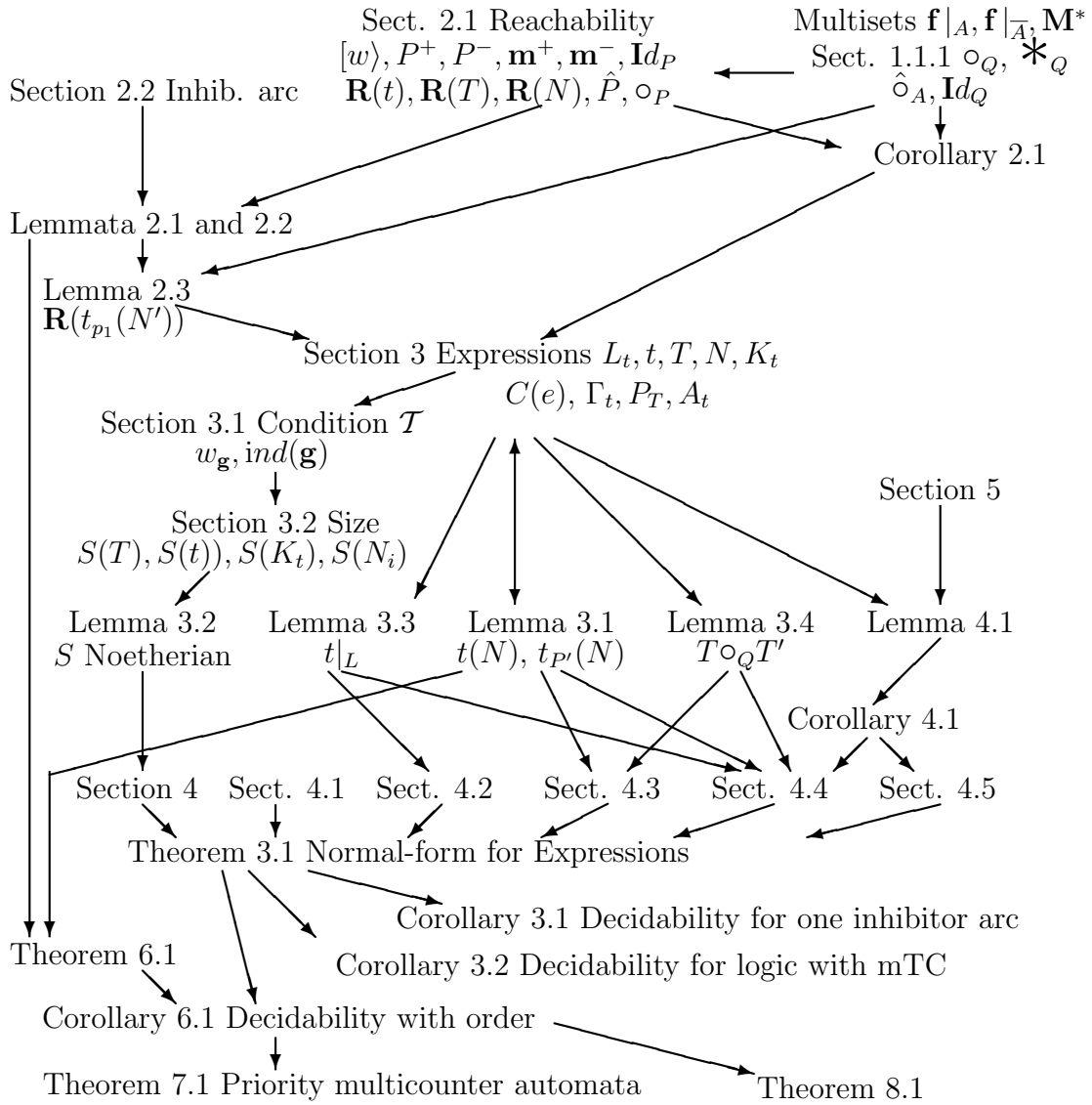


Figure 1: Map of dependencies and definitions in this paper

But the transitive closure over first order logic over natural numbers with the addition immediately becomes undecidable. For that reason the important principle of monotonicity in the reachability relation of Petri nets is combined with the idea of the transitive closure. That means the operator \star_Q is a monotone transitive closure (see Corollary 3.2). We consider the following three steps:

1. One application of \star_Q already allows us to express the reachability problem in a Petri net without inhibitor arcs (Corollary 2.1).
2. A second application of \star_Q (containing the first one in a nested way) allows us to express the reachability problem in a Petri net with one inhibitor arc (Lemma 2.3).
3. Arbitrary nested applications of \star_Q allow us to express the reachability problem in a Petri net for which there exists an ordering of the places such that a place has an inhibitor arc to all those transitions which have an inhibitor arc from a preceding place (Theorem 6.1).

In Section 3 we use expressions consisting of the operators \cup , \circ_Q and \star_Q on sets of multisets in a special form (Lemmata 3.1 and 3.4 show that we can bring every such expression in this form), which models the idea of a nested Petri net: The firing behavior of a complex (nested) transition is linked to firing sequences in inner Petri nets by a semilinear relation (unlike in the structured nets in [CK86]). The connection between these inner Petri nets corresponds to the chain of vector addition systems used in [Kos84] and it is described by the same semilinear relation. The main difference to the structure of the proofs in [Kos84] and [Lam92] is that states are not anymore necessary since their function is instead fulfilled (Section 4.4) by the nestedness of expressions (like regular expression replace a finite automaton).

Furthermore we define a condition (normal form \mathcal{T} corresponding to the property Θ in [Kos84]), which allows to check the emptiness of the expressed set of multisets, we define a size of the expressions leading to a Noetherian order and construct an algorithm in Section 4 which finds an equivalent expression fulfilling condition \mathcal{T} . Each step of the algorithm constructs an equivalent expression which is smaller with respect to the defined size. Some kind of pumping property (Lemma 4.1) which ensures the existence of arbitrary high firing sequences if condition \mathcal{T} is fulfilled is proved in Section 5.

The most general result (see Step 3 above) is described in Section 6 This allows to decide the expressed reachability problem. Sections 7 and 8 describe the conclusions for emptiness problems for automata.

An overview over the dependencies in this chapter is given in Figure 1. The end of the appendix contains a repetition of all definitions. The contents of this paper mainly corresponds to Chapter 5 in the habilitation-thesis [Rei05].

1.1 Multisets

For the sake of a flexible description, we use multi-sets instead of vectors. A *multi-set* over B is a function in \mathbb{N}^B .

We might write a multiset $\mathbf{f} \in \mathbb{N}^B$ as a set $\{b \mapsto f(b) \mid b \in B\}$, as a table

$$\left[\begin{array}{cccc} b_1 & b_2 & \dots & b_n \\ f(b_1) & f(b_2) & \dots & f(b_n) \end{array} \right] \text{ or as an } n\text{-ary vector } \begin{pmatrix} \mathbf{f}(b_1) \\ \mathbf{f}(b_2) \\ \vdots \\ \mathbf{f}(b_n) \end{pmatrix}. \text{ For the latter, we have to}$$

assume an ordering on $B = \{b_1, b_2, \dots, b_n\}$ (without relevance to the contents), and in the first two descriptions, we only need to write those b 's with $f(b) > 0$. Although we do not a priori limit the size of B , we only use multisets for a finite B in this paper. For multisets, we use the variables $\mathbf{c}, \mathbf{d}, \mathbf{e}, \mathbf{f}, \mathbf{g}, \mathbf{h}, \mathbf{m}, \mathbf{n}, \mathbf{r}, \mathbf{s}, \mathbf{x}, \mathbf{y}$, and for sets of multisets, we use the capitals $\mathbf{E}, \mathbf{L}, \mathbf{M}, \mathbf{N}, \mathbf{R}$ and \mathbf{Id} (the latter will denote the identity for the operator \circ_Q to be defined).

For $A \subseteq B$, we regard functions in $\mathbb{N}^A \subseteq \mathbb{N}^B$ as extended to zero for undefined values. This allows us to add any two multisets $\mathbf{f} \in \mathbb{N}^A$ and $\mathbf{g} \in \mathbb{N}^B$ and obtain a multiset in $(\mathbf{f} + \mathbf{g}) \in \mathbb{N}^{A \cup B}$ with $(\mathbf{f} + \mathbf{g})(x) = \mathbf{f}(x) + \mathbf{g}(x)$ in the same way as we would add the corresponding vectors assuming an ordering on $A \cup B$. The neutral element for addition is \emptyset with $\emptyset(x) = 0$ for all x . It holds $\mathbb{N}^A \cap \mathbb{N}^B = \mathbb{N}^{A \cap B}$.

The restriction $\mathbf{f}|_A$ of a multi-set $\mathbf{f} \in \mathbb{N}^B$ to A is

$$\mathbf{f}|_A(b) := \mathbf{f}(b) \text{ if } b \in A \text{ else } \mathbf{f}|_A(b) := 0.$$

This means $\mathbf{f}|_A := \{b \mapsto \mathbf{f}(b) \mid b \in A\}$. The complement operator is $\mathbf{f}|_{\bar{A}} := \{b \mapsto \mathbf{f}(b) \mid b \notin A\}$, thus $\mathbf{f} = \mathbf{f}|_A + \mathbf{f}|_{\bar{A}}$.

For a finite set $\mathbf{M} = \{\mathbf{m}_1, \dots, \mathbf{m}_k\} \subseteq \mathbb{N}^A$ of multi-sets,

$$\mathbf{M}^* := \{a_1 \mathbf{m}_1 + \dots + a_k \mathbf{m}_k \mid \forall i \leq k \ a_i \in \mathbb{N}\}$$

is the set of all linear combinations generated by \mathbf{M} . More generally, by $\mathbf{M}^0 := \{\emptyset\}$ and $\mathbf{M}^{i+1} := \mathbf{M}^i + \mathbf{M}$, we can define $\mathbf{M}^* := \bigcup_i \mathbf{M}^i$.

1.1.1 New operator on multisets

For an unambiguous¹ and injective binary relation Q , we define the operator \circ_Q on two sets of Multisets \mathbf{M} and \mathbf{N} as

$$\mathbf{N} \circ_Q \mathbf{M} := \left\{ \mathbf{n} \Big|_{\pi_1(Q)} + \mathbf{m} \Big|_{\pi_2(Q)} \mid \mathbf{n} \in \mathbf{N}, \mathbf{m} \in \mathbf{M}, \forall (a, b) \in Q \ \mathbf{n}(a) = \mathbf{m}(b) \right\}.$$

This means if \mathbf{n} and \mathbf{m} “match” according to Q , then the values for an $a \in \pi_1(Q) = \{a \mid (a, b) \in Q\}$ in \mathbf{n} and the values for a $b \in \pi_2(Q) = \{b \mid (a, b) \in Q\}$ in \mathbf{m} are “used up against each other” and the rest is added. For example,

$$\left\{ \left(\begin{array}{c} 3 \\ 6 \\ 1 \end{array} \right), \left(\begin{array}{c} 2 \\ 5 \\ 2 \end{array} \right) \right\} \circ_{\{(b_1, b_2)\}} \left\{ \left(\begin{array}{c} 8 \\ 3 \\ 1 \end{array} \right), \left(\begin{array}{c} 7 \\ 2 \\ 2 \end{array} \right), \left(\begin{array}{c} 5 \\ 2 \\ 3 \end{array} \right) \right\} = \left\{ \left(\begin{array}{c} 8 \\ 6 \\ 2 \end{array} \right), \left(\begin{array}{c} 7 \\ 5 \\ 4 \end{array} \right), \left(\begin{array}{c} 5 \\ 5 \\ 5 \end{array} \right) \right\}$$

¹A binary Q is unambiguous if Q^{-1} is injective.

or

$$\left\{ \left(\begin{array}{c} 3 \\ 6 \\ 1 \end{array} \right), \left(\begin{array}{c} 2 \\ 5 \\ 2 \end{array} \right) \right\} \circ_{\{(b_3, b_3)\}} \left\{ \left(\begin{array}{c} 8 \\ 3 \\ 1 \end{array} \right), \left(\begin{array}{c} 7 \\ 2 \\ 2 \end{array} \right), \left(\begin{array}{c} 5 \\ 2 \\ 3 \end{array} \right) \right\} = \left\{ \left(\begin{array}{c} 11 \\ 9 \end{array} \right), \left(\begin{array}{c} 9 \\ 7 \end{array} \right) \right\}$$

The latter example shows that the dimension is necessarily reduced (b_3 is used up on both sides) if $\pi_1(Q) \cap \pi_2(Q)$ is not empty. We will later need $\hat{\circ}_A := \circ_{\{(a,a)|a \in A\}}$ to get $\mathbf{N} \hat{\circ}_A \mathbf{M} = \{\mathbf{n}|_{\bar{A}} + \mathbf{m}|_{\bar{A}} \mid \mathbf{n} \in \mathbf{N}, \mathbf{m} \in \mathbf{M}, \forall a \in A \mathbf{n}(a) = \mathbf{m}(a)\}$. For example $\circ_{\{(b_3, b_3)\}} = \hat{\circ}_{\{b_3\}}$.

If $\pi_1(Q)$ and $\pi_2(Q)$ are disjoint, we define $\mathbf{Id}_Q := \{\{a \mapsto 1, b \mapsto 1\} \mid (a, b) \in Q\}^*$ which is the neutral element for \circ_Q . Obviously, it holds $\mathbf{N} \circ_{\emptyset} \mathbf{M} = \mathbf{N} + \mathbf{M}$ which makes $+$ with the neutral element $\mathbf{Id}_{\emptyset} = \{\emptyset\}$ a special case of the \circ_Q operator.

Furthermore, for Q with $\pi_1(Q)$ and $\pi_2(Q)$ disjoint, we define $\mathbf{*}_Q^0(\mathbf{M}) := \mathbf{Id}_Q$, $\mathbf{*}_Q^{i+1}(\mathbf{M}) := \mathbf{*}_Q^i(\mathbf{M}) \circ_Q (\mathbf{M} + \mathbf{Id}_Q)$ and $\mathbf{*}_Q(\mathbf{M}) := \bigcup_i \mathbf{*}_Q^i(\mathbf{M})$. Again, $\mathbf{*}_{\emptyset}(\mathbf{M}) = \mathbf{M}^*$ is a special case.

It is easy to see that $\mathbf{*}_Q(\mathbf{M})$ is the closure of $\mathbf{M} \cup \mathbf{Id}_Q$ under \circ_Q and the addition \circ_{\emptyset} :

If $\mathbf{f}, \mathbf{g} \in \mathbf{*}_Q(\mathbf{M})$ then there are $i, j \in \mathbb{N}$ with $\mathbf{f} \in \mathbf{*}_Q^i(\mathbf{M})$ and $\mathbf{g} \in \mathbf{*}_Q^j(\mathbf{M})$ thus $\{\mathbf{f}\} \circ_Q \{\mathbf{g}\} \subseteq \mathbf{*}_Q^i(\mathbf{M}) \circ_Q \mathbf{*}_Q^j(\mathbf{M}) = \mathbf{*}_Q^{i+j}(\mathbf{M}) \subseteq \mathbf{*}_Q(\mathbf{M})$.

Let $\mathbf{f}' = \mathbf{f} + \{a \mapsto \mathbf{g}(b), b \mapsto \mathbf{g}(b) \mid (a, b) \in Q\} \in \mathbf{f} + \mathbf{Id}_Q$ and $\mathbf{g}' = \mathbf{g} + \{a \mapsto \mathbf{f}(a), b \mapsto \mathbf{f}(a) \mid (a, b) \in Q\} \in \mathbf{g} + \mathbf{Id}_Q$. Then $\mathbf{f} + \mathbf{g} = \mathbf{f}'|_{\pi_1(Q)} + \mathbf{g}'|_{\pi_2(Q)} \in \{\mathbf{f}'\} \circ_Q \{\mathbf{g}'\} \subseteq \mathbf{*}_Q^i(\mathbf{M}) \circ_Q \mathbf{*}_Q^j(\mathbf{M}) = \mathbf{*}_Q^{i+j}(\mathbf{M}) \subseteq \mathbf{*}_Q(\mathbf{M})$ according to the definition of \circ_Q . Clearly $\mathbf{*}_Q(\mathbf{M}) = \mathbf{*}_Q(\mathbf{M}) + \mathbf{Id}_Q$.

For example, for $i > 0$ we have

$$\begin{aligned} \mathbf{*}_{\{(b_1, b_2)\}}^i \left(\left\{ \left(\begin{array}{c} 2 \\ 5 \\ 1 \end{array} \right) \right\} \right) &= \left\{ \left(\begin{array}{c} 2+j \\ 2+3i+j \\ i \end{array} \right) \mid j \geq 0 \right\} \text{ leading to} \\ \mathbf{*}_{\{(b_1, b_2)\}}^{i+1} \left(\left\{ \left(\begin{array}{c} 2 \\ 5 \\ 1 \end{array} \right) \right\} \right) &= \mathbf{*}_{\{(b_1, b_2)\}}^i \left(\left\{ \left(\begin{array}{c} 2 \\ 5 \\ 1 \end{array} \right) \right\} \right) \circ_{\{(b_1, b_2)\}} \left\{ \left(\begin{array}{c} 2+k \\ 5+k \\ 1 \end{array} \right) \mid k \geq 0 \right\} \\ &= \left\{ \left(\begin{array}{c} 2+j \\ 5+3i+j \\ i+1 \end{array} \right) \mid j \geq 0 \right\} \end{aligned}$$

(by “matching” with $k = 3i + j$) and

$$\mathbf{*}_{\{(b_1, b_2)\}} \left(\left\{ \left(\begin{array}{c} 2 \\ 5 \\ 1 \end{array} \right) \right\} \right) = \left\{ \left(\begin{array}{c} j \\ j \\ 0 \end{array} \right), \left(\begin{array}{c} 2+j \\ 2+3i+j \\ i \end{array} \right) \mid i, j \geq 0 \right\}$$

Remark: Adding \mathbf{Id}_Q is a crucial point: It corresponds to the monotonicity in Petri nets. Without this, deciding emptiness for the expressions would become undecidable.

1.1.2 Properties of the new operators

Obviously, it holds $\mathbf{N} \circ_Q \mathbf{M} = \mathbf{M} \circ_{Q^{-1}} \mathbf{N}$. Furthermore, we can express the intersection of $\mathbf{N}, \mathbf{M} \subseteq \mathbb{N}^A$ by $\mathbf{N} \circ_{Q'} \mathbf{L} \circ_{Q''} \mathbf{M} = \mathbf{N} \cap \mathbf{M}$ with $Q' := \{(a, a') \mid a \in A\}$, $Q'' := \{(a'', a) \mid a \in A\}$ and $\mathbf{L} := \{\{a \mapsto 1, a' \mapsto 1, a'' \mapsto 1\} \mid a \in A\}^*$.

Note here that, in general, $\mathbf{N} \circ_{Q'} \mathbf{L} \circ_{Q''} \mathbf{M}$ can only be written without brackets because $\pi_1(Q'') \cup (\{a \mid \exists \mathbf{f} \in \mathbf{M}, \mathbf{f}(a) > 0\} \setminus \pi_2(Q''))$ and $\pi_2(Q') \cup (\{a \mid \exists \mathbf{f} \in \mathbf{N}, \mathbf{f}(a) > 0\} \setminus \pi_1(Q'))$ are disjoint. If, additionally, $\pi_2(Q'')$ and $\{a \mid \exists \mathbf{f} \in \mathbf{N}, \mathbf{f}(a) > 0\}$ are disjoint and $\{a \mid \exists \mathbf{f} \in \mathbf{M}, \mathbf{f}(a) > 0\}$ and $\pi_1(Q')$ are disjoint, then $\mathbf{N} \circ_{Q'} \mathbf{L} \circ_{Q''} \mathbf{M} = \mathbf{L} \circ_{Q'^{-1} \cup Q''} (\mathbf{M} + \mathbf{N})$.

1.1.3 Semilinearity

The class of semilinear sets is the smallest class of sets of multisets containing all finite sets of multisets and being closed under $\cup, +$ and $*$. The semilinear sets are also closed under \cap , as shown in [GS65] and [ES69]. As a normal form to express semilinear sets, we will use the union of linear sets of the form $\mathbf{c} + \Gamma^*$ for a constant $\mathbf{c} \in \mathbb{N}^A$ and periods $\Gamma \subseteq \mathbb{N}^A$. (To improve efficiency of the algorithm described in this paper, we might demand that the union is disjoint; but efficiency is not an issue.)

The operator \circ_Q preserves semilinearity: Assume \mathbf{N} and \mathbf{M} are semilinear sets over A , then

$$\begin{aligned} \mathbf{N}' &:= \{\mathbf{f}' \mid \exists \mathbf{f} \in \mathbf{N} \quad \forall a \in \pi_1(Q) \quad \mathbf{f}'(a') = \mathbf{f}(a) \wedge \mathbf{f}'(a) = 0 \text{ and} \\ &\quad \forall a \notin \pi_1(Q) \quad \mathbf{f}'(a) = \mathbf{f}(a)\}, \\ \mathbf{M}' &:= \{\mathbf{f}' \mid \exists \mathbf{f} \in \mathbf{M} \quad \forall a \in \pi_2(Q) \quad \mathbf{f}'(a') = \mathbf{f}(a) \wedge \mathbf{f}'(a) = 0 \text{ and} \\ &\quad \forall a \notin \pi_2(Q) \quad \mathbf{f}'(a) = \mathbf{f}(a)\} \text{ and} \\ \mathbf{E}'_Q &:= \{\{a' \mapsto 1, b' \mapsto 1\}, \{c \mapsto 1\} \mid (a, b) \in Q, c \in A\}^* \\ &= \{\mathbf{f} \mid \forall (a, b) \in Q \quad \mathbf{f}(a') = \mathbf{f}(b')\} \end{aligned}$$

are as well semilinear sets over the set $A \cup \pi_1(Q)' \cup \pi_1(Q)'$ which is extended by new elements. Thus, $\mathbf{N} \circ_Q \mathbf{M} = ((\mathbf{N}' + \mathbf{M}') \cap \mathbf{E}'_Q) \Big|_{\overline{\pi_1(Q)' \cup \pi_1(Q)'}}$ is semilinear.

Since the closure under \cap is effective, this will allow a decision algorithm to

continue with the representation $\bigcup_{j=1}^l L_j$ by linear sets L_j .

On the other hand, \star_Q does not preserve semilinearity as the following example

shows: Let $\mathbf{M} := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}^*$, then $\star_{\{(b_3, b_2)\}}(\mathbf{M}) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \mid c \leq b2^a \right\}$ which

is not semilinear.

2 The reachability relation for Petri nets

2.1 The reachability relation for Petri nets without inhibitor arcs

We describe a *Petri net* as the triple $N = (P, T, W)$ with the places P , the transitions T and the weight function $W \in \mathbb{N}^{P \times T \cup T \times P}$. A transition $t \in T$ can fire from a marking $\mathbf{m} \in \mathbb{N}^P$ to a marking $\mathbf{m}' \in \mathbb{N}^P$, denoted by $\mathbf{m}[t]\mathbf{m}'$, if

$$\mathbf{m} - W(., t) = \mathbf{m}' - W(t, .) \in \mathbb{N}^P.$$

A *firing sequence* $w = t_1 \dots t_n \in T^*$ can fire from \mathbf{m}_0 to \mathbf{m}_n , denoted by $\mathbf{m}_0[w]\mathbf{m}_n$, if $\mathbf{m}_1, \dots, \mathbf{m}_{n-1}$ exist with $\mathbf{m}_0[t_1]\mathbf{m}_1[t_2]\dots[t_n]\mathbf{m}_n$. The *reachability problem* is to decide for a given net N with start- and end markings $\mathbf{m}_0, \mathbf{m}_e \in \mathbb{N}^P$, if there is a $w \in T^*$ with $\mathbf{m}_0[w]\mathbf{m}_e$.

Let $P^+ := \{p^+ \mid p \in P\}$ and $P^- := \{p^- \mid p \in P\}$ be copies of the places. For any multiset, \mathbf{m} we define the corresponding copies $\mathbf{m}^- := \{p^- \mapsto \mathbf{m}(p) \mid p \in P\}$ and $\mathbf{m}^+ := \{p^+ \mapsto \mathbf{m}(p) \mid p \in P\}$. Then, we can define the *reachability relation* for a transition t as

$$\begin{aligned} \mathbf{R}(t) &:= \{ \mathbf{m}^- + \mathbf{m}'^+ \mid \mathbf{m}[t]\mathbf{m}' \} \\ &= \left\{ \mathbf{r} \in \mathbb{N}^{P^+ \cup P^-} \mid \forall p \in P \mathbf{r}(p^-) - W(p, t) = \mathbf{r}(p^+) - W(t, p) \in \mathbb{N} \right\} \end{aligned}$$

and the reachability relation for a set of transitions T as $\mathbf{R}(T) := \bigcup_{t \in T} \mathbf{R}(t)$.

The important property of monotonicity means that whenever $\mathbf{m}[w]\mathbf{m}'$, then also $(\mathbf{m} + \mathbf{n})[w](\mathbf{m}' + \mathbf{n})$ for any $\mathbf{n} \in \mathbb{N}^P$. This corresponds to adding $\mathbf{Id}_P := \mathbf{Id}_{\hat{P}}$ with $\hat{P} := \{(p^+, p^-) \mid p \in P\}$ and $\mathbf{R}(t)$ can be written as the linear set $\mathbf{R}(t) = \mathbf{c}_t + \mathbf{Id}_P$ using \mathbf{c}_t with $\mathbf{c}_t(p^-) := W(p, t)$ and $\mathbf{c}_t(p^+) := W(t, p)$ for all $p \in P$. The reachability relation for the concatenation of two firing sequences is described by $\mathbf{R}(w_1 w_2) = \mathbf{R}(w_1) \circ_P \mathbf{R}(w_2)$ with $\circ_P := \circ_{\hat{P}}$ and the iteration is done by $\mathbf{*}_P := \mathbf{*}_{\hat{P}}$. We define the reachability relation of the Petri net N as $\mathbf{R}(N) := \mathbf{R}(T^*) := \mathbf{*}_P(\mathbf{R}(T))$. The reachability problem formulates as $(\mathbf{m}_0^- + \mathbf{m}_e^+) \in \mathbf{R}(N)$.

Corollary 2.1 *There is a firing sequence $w \in T^*$ with $\mathbf{m}_0[w]\mathbf{m}_e$ in N if and only if*

$$\mathbf{m}_0^+ \circ_P \mathbf{R}(N) \circ_P \mathbf{m}_e^- = (\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^-} \mathbf{R}(N) = \{\emptyset\}.$$

(If there is no $w \in T^$ with $\mathbf{m}_0[w]\mathbf{m}_e$, we have $(\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^-} \mathbf{R}(N) = \emptyset$.)*

Example:

Consider the following Petri net N with

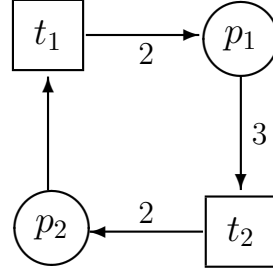
$$\mathbf{R}(t_1) = \{p_2^- \mapsto 1, p_1^+ \mapsto 2\} + \mathbf{Id}_P,$$

$$\mathbf{R}(t_2) = \{p_1^- \mapsto 3, p_2^+ \mapsto 2\} + \mathbf{Id}_P,$$

$$\text{and thus, } \mathbf{R}(T) = \left\{ \begin{bmatrix} p_2^- \\ p_1^+ \end{bmatrix}, \begin{bmatrix} p_1^- \\ p_2^+ \end{bmatrix} \right\}.$$

$$\text{Let furthermore } \mathbf{m}_0 = \{p_1 \mapsto 1, p_2 \mapsto 1\}$$

$$\text{and } \mathbf{m}_e = \{p_1 \mapsto 1, p_2 \mapsto 2\}.$$



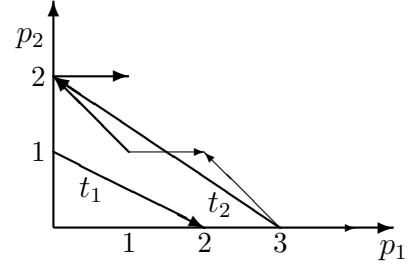
By concatenating, we get for example

$$\mathbf{R}(t_1 t_2) = \mathbf{R}(t_1) \circ_P \mathbf{R}(t_2) = \begin{bmatrix} p_2^- \\ p_1^- \\ p_2^+ \end{bmatrix} + \mathbf{Id}_P \text{ and}$$

$$\mathbf{R}(t_1 t_1 t_2) = \begin{bmatrix} p_2^- \\ p_2^+ \\ p_1^+ \end{bmatrix} + \mathbf{Id}_P.$$

$$\text{In this way we get } (\mathbf{m}_0^- + \mathbf{m}_e^+) = \begin{bmatrix} p_1^- \\ p_1^- \\ p_1^+ \\ p_2^+ \end{bmatrix}$$

$$\in \mathbf{R}(t_1 t_2 t_1 t_1 t_2) = \begin{bmatrix} p_1^- \\ p_1^- \\ p_1^+ \\ p_2^+ \end{bmatrix} + \mathbf{Id}_P \subseteq \mathbf{R}(N).$$



By further concatenating and iterating, we get

$$\mathbf{R}((t_1 t_2)^*) = \begin{bmatrix} p_2^- \\ p_1^+ \end{bmatrix} + \begin{bmatrix} p_1^- \\ p_2^+ \end{bmatrix}^* + \mathbf{Id}_P,$$

$$\mathbf{R}(t_2 t_1) = \mathbf{R}(t_2) \circ_P \mathbf{R}(t_1) = \begin{bmatrix} p_1^- \\ p_2^+ \\ p_1^+ \end{bmatrix} + \mathbf{Id}_P,$$

$$\mathbf{R}((t_2 t_1)^*) = \begin{bmatrix} p_1^- \\ p_2^+ \end{bmatrix} + \begin{bmatrix} p_1^- \\ p_2^+ \end{bmatrix}^* + \mathbf{Id}_P,$$

$$\mathbf{R}(t_2 t_1 t_1) = \begin{bmatrix} p_1^- \\ p_1^+ \end{bmatrix} + \mathbf{Id}_P,$$

$$\mathbf{R}((t_2 t_1 t_1)^*) = \begin{bmatrix} p_1^- \\ p_1^+ \end{bmatrix} + \begin{bmatrix} p_1^+ \end{bmatrix}^* + \mathbf{Id}_P,$$

$$\mathbf{R}((t_1 t_1 t_2)^*) = \begin{bmatrix} p_2^- \\ p_2^+ \end{bmatrix} + \begin{bmatrix} p_1^+ \end{bmatrix}^* + \mathbf{Id}_P,$$

$$\mathbf{R}((t_1 t_2 t_1)^*) = \begin{bmatrix} p_2^- \\ p_2^+ \\ p_1^- \\ p_2^+ \end{bmatrix} + \begin{bmatrix} p_1^+ \end{bmatrix}^* + \mathbf{Id}_P,$$

...

which finally yields $\mathbf{R}(N) = \mathbf{R}(T^*) = \mathbf{*}_P \left(\left\{ \begin{bmatrix} p_2^- \\ p_1^+ \end{bmatrix}, \begin{bmatrix} p_1^- \\ p_2^+ \end{bmatrix} \right\} \right) =$

$$\begin{aligned} & \left\{ \begin{bmatrix} p_2^- \\ p_1^+ \end{bmatrix}, \begin{bmatrix} p_1^- \\ p_2^+ \end{bmatrix} \right\}^* + \mathbf{Id}_P \cup \\ & \left\{ \begin{bmatrix} p_2^- \\ p_1^+ \end{bmatrix}, \begin{bmatrix} p_1^- \\ p_1^+ \end{bmatrix} \right\} + \left\{ \begin{bmatrix} p_1^- \\ p_2^+ \end{bmatrix}, \begin{bmatrix} p_2^- \\ p_1^+ \end{bmatrix}, \begin{bmatrix} p_1^- \\ p_2^+ \end{bmatrix} \right\}^* + \mathbf{Id}_P \cup \\ & \left\{ \begin{bmatrix} p_1^- \\ p_3^+ \end{bmatrix}, \begin{bmatrix} p_2^- \\ p_2^+ \end{bmatrix}, \begin{bmatrix} p_2^- \\ p_1^+ \\ p_1^- \\ p_2^+ \end{bmatrix}, \begin{bmatrix} p_1^- \\ p_3^+ \end{bmatrix}, \begin{bmatrix} p_2^- \\ p_2^+ \\ p_1^+ \end{bmatrix}, \begin{bmatrix} p_1^- \\ p_2^- \\ p_1^+ \end{bmatrix} \right\}, \\ & \left\{ \begin{bmatrix} p_2^- \\ p_1^+ \end{bmatrix} \right\} + \left\{ \begin{bmatrix} p_2^- \\ p_1^+ \end{bmatrix}, \begin{bmatrix} p_1^- \\ p_3^+ \end{bmatrix}, \begin{bmatrix} p_1^- \\ p_2^+ \end{bmatrix}, \begin{bmatrix} p_1^- \\ p_2^+ \end{bmatrix}, \begin{bmatrix} p_1^+ \end{bmatrix}, \begin{bmatrix} p_2^+ \end{bmatrix} \right\}^* + \mathbf{Id}_P \end{aligned}$$

2.2 Petri nets with inhibitor arcs

An *inhibitor arc* from a place to a transition means that the transition can only fire if no token is on that place. We describe such a Petri net as the 6-tuple $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ with the places P , the transitions T , the weight function $W \in \mathbb{N}^{P \times T \cup T \times P}$, the inhibitor arcs $I \subseteq P \times T$ and, the start and end markings $\mathbf{m}_0, \mathbf{m}_e \in \mathbb{N}^P$. We will denote an inhibitor arc in the pictures by $\text{---}\bullet$.

A transition $t \in T$ can fire from a marking $\mathbf{m} \in \mathbb{N}^P$ to a marking $\mathbf{m}' \in \mathbb{N}^P$, denoted by $\mathbf{m}[t]\mathbf{m}'$ if

$$\mathbf{m} - W(., t) = \mathbf{m}' - W(t, .) \in \mathbb{N}^P \text{ and } \forall p \in P (p, t) \in I \rightarrow \mathbf{m}(p) = 0.$$

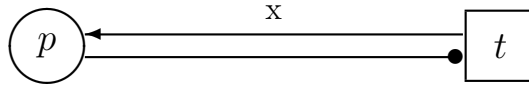
A firing sequence $w = t_1 \dots t_n \in T^*$ can fire from \mathbf{m}_0 to \mathbf{m}_n , denoted by $\mathbf{m}_0[w]\mathbf{m}_n$, if there exist intermediate markings $\mathbf{m}_1, \dots, \mathbf{m}_{n-1}$ with $\mathbf{m}_0[t_1]\mathbf{m}_1[t_2]\dots[t_n]\mathbf{m}_n$.

The *reachability problem* for a Petri net $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ is to decide, whether there exists a $w \in T^*$ with $\mathbf{m}_0[w]\mathbf{m}_e$.

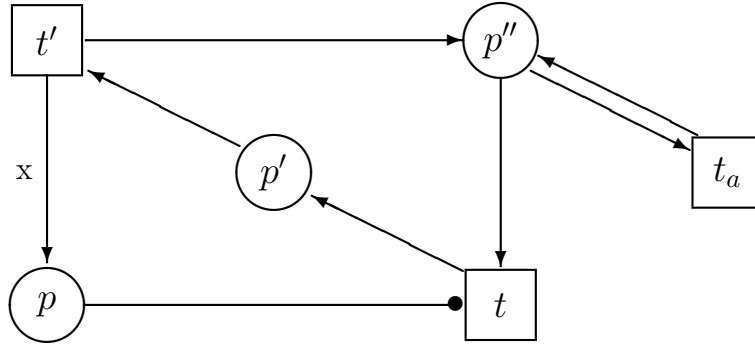
In the following two lemmata, we restrict the cases for which we have to regard the reachability problem. The aim of the first lemma is to make the reachability problem symmetric, that means the reachability problem is the same for $(P, T, W^{-1}, I, \mathbf{m}_e, \mathbf{m}_0)$ with $W^{-1} := \{(x, y) \mid (y, x) \in W\}$:

Lemma 2.1 *Each Petri net $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ can be changed in such a way that the condition $\forall p \in P, t \in T (p, t) \in I \rightarrow W(t, p) = 0$ holds without changing the inhibitor arcs or the reachability problem.*

Proof: Consider a transition $t \in T$ such that there exists a $p \in P$ with $(p, t) \in I$ and $W(t, p) = x > 0$. This is depicted by



We add a new transition t' in $T' := T \cup \{t'\}$ and two new places p' and p'' in $P' := P \cup \{p', p''\}$. Furthermore, we put an additional token on p'' in the start-marking $\mathbf{m}'_0 := \mathbf{m}_0 + \{p'' \mapsto 1\}$ and the end-marking $\mathbf{m}'_e := \mathbf{m}_e + \{p'' \mapsto 1\}$. Set $W'(t', .) := W(t, .) + \{p'' \mapsto 1\}$ which means that all the arcs from the transition t and an arc to p'' are now arcs from the transition t' . An arc from p'' to t is then added, which means $W'(., t) := W(., t) + \{p'' \mapsto 1\}$. Set $W'(t, .) := W'(., t') := \{p' \mapsto 1\}$, $W'(t_a, .) := W(t_a, .) + \{p'' \mapsto 1\}$ and $W'(., t_a) := W(., t_a) + \{p'' \mapsto 1\}$ for every $t_a \in T \setminus \{t\}$.



There will always be exactly one token on either p' or p'' . If t fires, then no token is on p'' and so t' is the only transition which can fire. The firing of tt' (together) has the same effect on the net as the firing of t before the change; hence, the reachability problem remains the same. ■

A general aim of the decision algorithm explained below is to reduce the number of places and transitions and, therefore, transfer the information to a structural description. However, in the next lemma we do a step in the opposite direction in order to make the description of the reachability relation easier.

Lemma 2.2 *Each Petri net $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ can be changed in a way such that the condition $\forall p \in P, t \in T (p, t) \in I \rightarrow \mathbf{m}_0(p) = \mathbf{m}_e(p) = 0$ holds by changing neither the inhibitor arcs, the condition in Lemma 2.1 nor the reachability problem.*

Proof: We add two new transitions t and t' in $T' := T \cup \{t, t'\}$, and three new places p, p' and p'' in $P' := P \cup \{p, p', p''\}$. Set $W'(t, \cdot) := \mathbf{m}_0 + \{p' \mapsto 1\}$, $W'(\cdot, t') := \mathbf{m}_e + \{p' \mapsto 1\}$, $W'(\cdot, t) := \{p \mapsto 1\}$, $W'(t', \cdot) := \{p'' \mapsto 1\}$, $\mathbf{m}'_0 := \{p \mapsto 1\}$ and $\mathbf{m}'_e := \{p'' \mapsto 1\}$. For every $t_a \in T$, we set $W'(t_a, \cdot) := W(t_a, \cdot) + \{p' \mapsto 1\}$ and $W'(\cdot, t_a) := W(\cdot, t_a) + \{p' \mapsto 1\}$. This prevents a firing before t and after t' . Therefore, t is the first and t' is the last transition to fire, but they can only fire once. Obviously, the reachability problem from the marking after the firing of t to the marking before the firing of t' is the same as before. ■

2.3 The reachability relation for Petri nets with one inhibitor arc

Let us consider a Petri-net $N = (P, T, W, \{(p_1, \hat{t})\}, \mathbf{m}_0, \mathbf{m}_e)$ having the property of lemmata 2.1 and 2.2. As in the case of no inhibitor arcs, we can describe by $\mathbf{R}(N') = \mathbf{K}_P(\mathbf{R}(T \setminus \{\hat{t}\}))$ for the Petri net $N' = (P, T \setminus \{\hat{t}\}, W|_{\overline{P \times \{\hat{t}\} \cup \{\hat{t}\} \times P}})$ the reachability relation for firing sequences $w \in (T \setminus \{\hat{t}\})^*$. The restriction to those firing sequences starting and ending with markings without tokens on p_1 can be described by $\mathbf{R}(t_{p_1}(N')) := \{\mathbf{r} \in \mathbf{R}(N') \mid \mathbf{r}(p_1^-) = \mathbf{r}(p_1^+) = 0\}$. In $\mathbf{R}(N) := \mathbf{K}_{P \setminus \{p_1\}}(\mathbf{R}(t_{p_1}(N')) \cup \mathbf{R}(\hat{t}))$ with $\mathbf{R}(\hat{t}) = \mathbf{c}_i + \mathbf{Id}_{P \setminus \{p_1\}}$ we iterate these

parts together with the alternative of using \hat{t} . Generalizing Corollary 2.1 we get the following:

Lemma 2.3 *Given a Petri-net $N = (P, T, W, \{(p_1, \hat{t})\}, \mathbf{m}_0, \mathbf{m}_e)$ with only one inhibitor arc (p_1, \hat{t}) having the property of lemmata 2.1 and 2.2, then there is a firing sequence $w \in T^*$ with $\mathbf{m}_0[w]\mathbf{m}_e$ if and only if*

$$\mathbf{m}_0^+ \circ_{P \setminus \{p_1\}} \mathbf{R}(N) \circ_{P \setminus \{p_1\}} \mathbf{m}_e^- = (\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^- \setminus \{p_1^+, p_1^-\}} \mathbf{R}(N) = \{\emptyset\}.$$

(If there is no such $w \in T^*$, we have $(\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^- \setminus \{p_1^+, p_1^-\}} \mathbf{R}(N) = \emptyset$.)

Proof: A firing sequence $w \in T^*$ can be decomposed in minimal firing sequences $w_1 \dots w_k = w$ having the property $\mathbf{m}_0[w_1]\mathbf{m}_1[w_2] \dots [w_k]\mathbf{m}_k$ with $\mathbf{m}_k = \mathbf{m}_e$ such that $\mathbf{m}_i(p_1) = 0$ for all $i \leq k$.

Each w_i is either equal to \hat{t} or in $(T \setminus \{\hat{t}\})^*$. This holds since the occurrence of \hat{t} in a w_i with $|w_i| > 1$ would mean that, at that time during the firing of w_i , there is no token on p_1 , and thus, w_i would not be minimal.

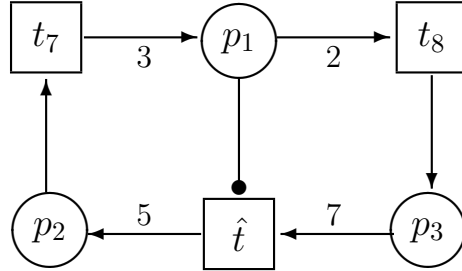
If $w_i \in (T \setminus \{\hat{t}\})^*$ then $\mathbf{m}_{i-1}^- + \mathbf{m}_i^+ \in \mathbf{R}(N')$. Then from $\mathbf{m}_{i-1}(p_1) = 0$ and $\mathbf{m}_i(p_1) = 0$ it follows that $\mathbf{m}_{i-1}^- + \mathbf{m}_i^+ \in \mathbf{R}(t_{p_1}(N'))$. Otherwise, if $w_i = \hat{t}$, we have $\mathbf{m}_{i-1}^- + \mathbf{m}_i^+$ in $\mathbf{R}(\hat{t})$.

Concatenating all with the operator $\star_{P \setminus \{p_1\}}$ leads to $\mathbf{m}_0^- + \mathbf{m}_e^+$ is in $\mathbf{R}(N)$, which means $(\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^-} \mathbf{R}(N) = \{\emptyset\}$.

The other direction follows simply by composing firing sequences. ■

Example:

Consider the Petri net N



with the start marking $\{p_2 \mapsto 4, p_3 \mapsto 2\}$ and the end marking $\{p_2 \mapsto 4, p_3 \mapsto 3\}$. We have $\mathbf{R}(t_7) = \{p_2^- \mapsto 1, p_1^+ \mapsto 3\} + \mathbf{Id}_P$, $\mathbf{R}(t_8) = \{p_1^- \mapsto 2, p_3^+ \mapsto 1\} + \mathbf{Id}_P$ and $\mathbf{R}(\hat{t}) = \{p_3^- \mapsto 7, p_2^+ \mapsto 5\} + \mathbf{Id}_{P \setminus \{p_1\}}$. This yields

$$\begin{aligned} \mathbf{R}(N') &= \mathbf{R}((P, \{t_7, t_8\}, W |_{P \times \{\hat{t}\} \cup \{\hat{t}\} \times P})) = \star_P \left(\left\{ \left[\begin{array}{cc} p_2^- & p_1^+ \\ 1 & 3 \end{array} \right], \left[\begin{array}{cc} p_1^- & p_3^+ \\ 2 & 1 \end{array} \right] \right\} \right) = \\ & \left\{ \left[\begin{array}{cc} p_2^- & p_1^+ \\ 1 & 3 \end{array} \right], \left[\begin{array}{cc} p_1^- & p_3^+ \\ 2 & 1 \end{array} \right], \left[\begin{array}{ccc} p_2^- & p_1^+ & p_3^+ \\ 1 & 1 & 1 \end{array} \right], \left[\begin{array}{ccc} p_2^- & p_1^- & p_3^+ \\ 1 & 1 & 2 \end{array} \right], \left[\begin{array}{ccc} p_2^- & p_1^+ & p_3^+ \\ 2 & 2 & 2 \end{array} \right], \left[\begin{array}{ccc} p_2^- & p_3^+ \\ 2 & 3 \end{array} \right] \right\}^* + \mathbf{Id}_P \\ \text{and } \mathbf{R}(t_{p_1}(N')) &= \mathbf{R}(N') \circ_{\{(p_1^-, x), (p_1^+, y)\}} \{\emptyset\} = \left\{ \left[\begin{array}{cc} p_2^- & p_3^+ \\ 2 & 3 \end{array} \right] \right\}^* + \mathbf{Id}_{\{p_2, p_3\}}. \end{aligned}$$

We can cut the firing sequences in $(t_7 + t_8 + \hat{t})^* = ((t_7 + t_8)^* + \hat{t})^*$ into parts in $(t_7 + t_8)^*$ and \hat{t} all starting and ending with no token on p_1 . This yields

$$\begin{aligned} \mathbf{R}(N) &:= \star_{\{p_2, p_3\}} (\mathbf{R}(t_{p_1}(N')) \cup \mathbf{R}(\hat{t})) \\ & \left\{ \left[\begin{array}{cc} p_2^- & p_3^+ \\ 2 & 3 \end{array} \right], \left[\begin{array}{cc} p_3^- & p_2^+ \\ 7 & 5 \end{array} \right], \left[\begin{array}{ccc} p_2^- & p_3^- & p_2^+ \\ 2 & 4 & 5 \end{array} \right], \left[\begin{array}{ccc} p_2^- & p_3^- & p_2^+ \\ 4 & 1 & 5 \end{array} \right], \left[\begin{array}{ccc} p_3^- & p_2^+ & p_3^+ \\ 7 & 3 & 3 \end{array} \right], \left[\begin{array}{ccc} p_3^- & p_2^+ & p_3^+ \\ 7 & 1 & 6 \end{array} \right], \dots, \\ & \left[\begin{array}{ccc} p_2^- & p_3^- & p_2^+ \\ 4 & 2 & 8 \end{array} \right], \left[\begin{array}{ccc} p_2^- & p_3^- & p_2^+ \\ 5 & 1 & 7 \end{array} \right], \left[\begin{array}{ccc} p_3^- & p_2^+ & p_3^+ \\ 6 & 4 & 3 \end{array} \right], \left[\begin{array}{ccc} p_2^- & p_3^- & p_2^+ \\ 4 & 2 & 4 \end{array} \right], \left[\begin{array}{ccc} p_2^- & p_3^- & p_2^+ \\ 4 & 4 & 3 \end{array} \right] \right\}^* + \mathbf{Id}_{\{p_2, p_3\}}. \end{aligned}$$

3 Nested Petri Nets as normal form for expressions

From now on we use the variables t, T, N as expressions describing transitions, sets of transitions and (sub-)nets. For an expression e , we will always define a *carrier set* $C(e) \supseteq \{a \mid \exists \mathbf{f} \in \mathbf{R}(e), \mathbf{f}(a) > 0\}$. The function \mathbf{R} was in the previous section giving the reachability relation $\mathbf{R}(e) \subseteq \mathbb{N}^{C(e)}$ for an e of the form t, N or T . Now, we use \mathbf{R} as the evaluation function for an expression where the expression operators $\star_{P, \circ_Q, \cup}$ and $+$, and the operator \cap will always be defined on expressions such that they commute with \mathbf{R} .

Let the expression for an *elementary transition* have the form $t = L_t$, where L_t is an expression for the *linear set* $\mathbf{L}_t = \mathbf{R}(L_t) = \mathbf{c}_t + \Gamma_t^*$ described by a (constant) multiset \mathbf{c}_t and a finite set of (period-) multisets Γ_t . For example, in Sections 2.1 and 2.3, we have $\Gamma_t = \{\{p^- \mapsto 1, p^+ \mapsto 1\} \mid p \in P\}$ leading to $\Gamma_t^* = \text{Id}_P$. We have $C(t) := C(L_t) := P^- \cup P^+ \cup \{a \mid \exists \mathbf{f} \in \{c_t\} \cup \Gamma_t, \mathbf{f}(a) > 0\}$.

Let the expression for *sets of transitions* be $T = t_1 \cup t_2 \dots \cup t_l$ for expressions for transitions $t_i \in T$ for $i \leq l$, and the expression for a *sub-net* with places P_T and transitions T be $N = \star_{P_T}(T)$. Let $C(N) := C(T) := \bigcup_{t \in T} C(t)$.

Let the expression for a *generalized transition* have the form $t = L_t \hat{\circ}_{A_t} K_t$, where L_t again expresses a linear set, and K_t is a set of sub-nets and interpreted as expression $K_t = \sum_{N_i \in K_t} N_i$ where the $C(N_i)$ are pairwise disjoint and

$$A_t := C(K_t) := \bigcup_{N_i \in K_t} C(N_i).$$

We define $C(t) := \{a \mid \exists \mathbf{f} \in \{c_t\} \cup \Gamma_t, \mathbf{f}(a) > 0\} \setminus A_t$. This means that the behavior of t is mainly described by the linear set $\mathbf{c}_t + \Gamma_t^* \subseteq \mathbb{N}^{C(t) \cup A_t}$ but it is additionally controlled by the reachability in the sub-nets N_i .

For example, the reachability question for a complete net in Corollary 2.1 and Lemma 2.3 is formulated as the control by a (sub-)net in t with $\mathbf{R}(t) = (\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^-} \mathbf{R}(N)$ respectively $\mathbf{R}(t) = (\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P^+ \cup P^- \setminus \{p_1^+, p_1^-\}} \mathbf{R}(N)$ in a way that already has this normal form for expressions with $c_t = (\mathbf{m}_0^- + \mathbf{m}_e^+)$ and $\Gamma_t = \emptyset$. However, the behavior on the outside is trivial (\emptyset or $\{\emptyset\}$) because $C(t) = \emptyset$ in this case. The same holds for $(\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P_{T_{g-1}}^+ \cup P_{T_{g-1}}^-} \star_{P_{T_{g-1}}}(T_{g-1})$ in Theorem 6.1.

The decision algorithm in Section 4 will start with an expression of the form $T = \{t\}$ by keeping in mind that, according to Lemma 2.3, $\mathbf{R}(T) = \mathbf{R}(t) = \{\emptyset\}$ if there is a firing sequence $w \in T^*$ with $\mathbf{m}_0[w] \mathbf{m}_e$. Otherwise $\mathbf{R}(T) = \mathbf{R}(t) = \emptyset$ if there is not.

An example for the general case is $\mathbf{R}(t_{p_1}(N')) := \{\mathbf{r} \in \mathbf{R}(N') \mid \mathbf{r}(p_1^-) = \mathbf{r}(p_1^+) = 0\}$ from Section 2.3. Here we need Lemma 3.1 to construct the expression $t_{\{p_1\}}(N')$ of the desired normal form with $\mathbf{R}(t_{p_1}(N')) = \mathbf{R}(t_{\{p_1\}}(N'))$. Again, t_h in Theorem 6.1 has the normal form for expressions.

Lemma 3.1 *Let N be an expression for a subnet. Then, we can construct an equivalent expression for a transition $t(N)$ with $\mathbf{R}(t(N)) = \mathbf{R}(N)$ and $t_{P'}(N)$ with $\mathbf{R}(t_{P'}(N)) = \{\mathbf{m} \in \mathbf{R}(N) \mid \forall p \in P' \mathbf{m}(p^-) = \mathbf{m}(p^+) = 0\}$.*

Proof: Define $t(N)$ by $\mathbf{c}_{t(N)} := \mathbf{c}_{t_{P'}(N)} := \emptyset$, $\Gamma_{t(N)} := \{\{q \mapsto 1, \hat{q} \mapsto 1\} \mid q \in C(N)\}$ and $K_{t(N)} := \{\hat{N}\}$ where \hat{N} is the result of replacing all occurrences of some $q \in C(N)$ in N by \hat{q} . This means that we make the $C(\hat{N})$ disjoint to $C(t(N))$.

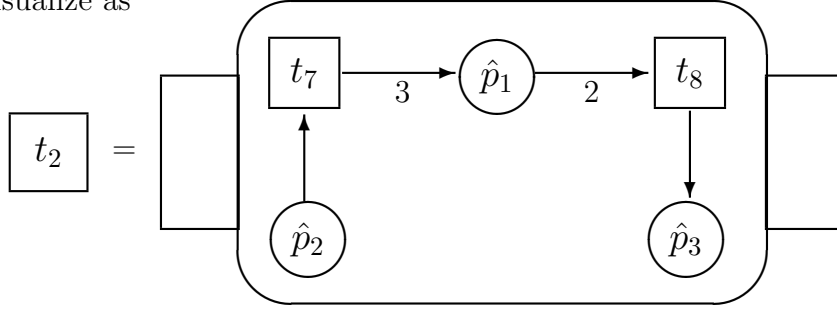
The restriction of places in P' to 0 is done by $\Gamma_{t_{P'}(N)} := \{\mathbf{m} \in \Gamma_{t(N)} \mid \forall p \in P' \mathbf{m}(p^-) = \mathbf{m}(p^+) = 0\}$. ■

Example (continued):

We identify $t_7 = \{\hat{p}_2^- \mapsto 1, \hat{p}_1^+ \mapsto 3\} + \mathbf{Id}_{\{\hat{p}_1, \hat{p}_2, \hat{p}_3\}}$, $t_8 = \{\hat{p}_1^- \mapsto 2, \hat{p}_3^+ \mapsto 1\} + \mathbf{Id}_{\{\hat{p}_1, \hat{p}_2, \hat{p}_3\}}$ and $\hat{t} = \{p_3^- \mapsto 7, p_2^+ \mapsto 5\} + \mathbf{Id}_{\{p_2, p_3\}}$. This yields the expressions $T_1 = t_7 \cup t_8$ and $N_1 = \ast_{\{\hat{p}_1, \hat{p}_2, \hat{p}_3\}}(T_1)$. On the next level, we get the generalized transition $t_2 = t_{p_1}(N')$

$$\left(\emptyset + \left\{ \left[\begin{smallmatrix} p_2^- \\ 1 \end{smallmatrix}, \begin{smallmatrix} \hat{p}_2^- \\ 1 \end{smallmatrix} \right], \left[\begin{smallmatrix} p_3^- \\ 1 \end{smallmatrix}, \begin{smallmatrix} \hat{p}_3^- \\ 1 \end{smallmatrix} \right], \left[\begin{smallmatrix} p_2^+ \\ 1 \end{smallmatrix}, \begin{smallmatrix} \hat{p}_2^+ \\ 1 \end{smallmatrix} \right], \left[\begin{smallmatrix} p_3^+ \\ 1 \end{smallmatrix}, \begin{smallmatrix} \hat{p}_3^+ \\ 1 \end{smallmatrix} \right] \right\} \right) \hat{\circ}_{\{\hat{p}_2^-, \hat{p}_3^-, \hat{p}_2^+, \hat{p}_3^+\}} N_1,$$

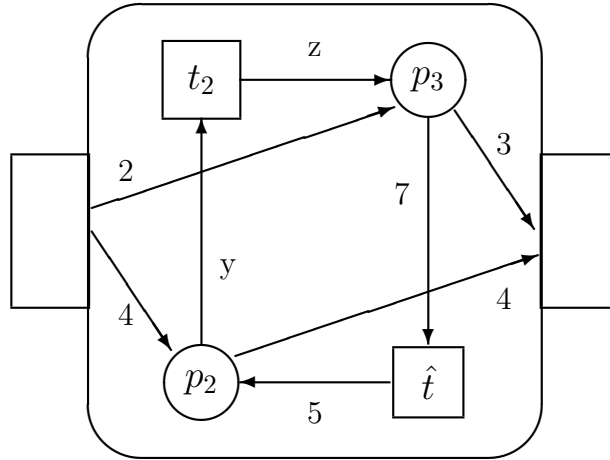
which we visualize as



$T_2 = t_2 \cup \hat{t}$ and $N_2 = \ast_{\{p_2, p_3\}}(T_2)$. On the top level, we get

$$T_3 = t_3 = \left[\begin{smallmatrix} p_2^- \\ 4 \end{smallmatrix}, \begin{smallmatrix} p_3^- \\ 2 \end{smallmatrix}, \begin{smallmatrix} p_2^+ \\ 4 \end{smallmatrix}, \begin{smallmatrix} p_3^+ \\ 3 \end{smallmatrix} \right] \hat{\circ}_{\{p_2^-, p_3^-, p_2^+, p_3^+\}} N_2,$$

which we visualize as follows:



3.1 The property \mathcal{T}

In order to decide the emptiness problem for expressions, we want to establish a normal form \mathcal{T} , which corresponds to the condition Θ in [Kos84]:

Definition An expression T has the property \mathcal{T} if $\forall t \in T, \forall N_i = \star_{P_{T_i}}(T_i) \in K_t$ the following conditions hold:

1. In recursive manner, T_i has
 - (a) the property \mathcal{T} , and
 - (b) For all $t' \in T_i$ it holds $\forall \mathbf{g} \in \{\mathbf{c}_{t'}\} \cup \Gamma_{t'} \exists w_{\mathbf{g}} \in C(t') \mathbf{g}(w_{\mathbf{g}}) = 1,$
 $\forall \mathbf{g}' \in \bigcup_{t' \in T_i} \{\mathbf{c}_{t'}\} \cup \Gamma_{t'} \setminus \{\mathbf{g}\} \mathbf{g}'(w_{\mathbf{g}}) = 0.$

This condition says that the number of times where \mathbf{g} is used is exactly the number of occurrence of the *witness (place)* $w_{\mathbf{g}}$.
2. $\forall \mathbf{g} \in \{\mathbf{c}_t\} \cup \Gamma_t, \forall p \in P_{T_i} \mathbf{g}(p^-) - \text{ind}(\mathbf{g})(p^-) = \mathbf{g}(p^+) - \text{ind}(\mathbf{g})(p^+)$, where

$$\text{ind}(\mathbf{g}) := \sum_{t' \in T_i, \mathbf{g}' \in \{\mathbf{c}_{t'}\} \cup \Gamma_{t'}} \mathbf{g}(w_{\mathbf{g}'}) \mathbf{g}'$$

describes the indirect effect of \mathbf{g} using the property about the witness places in Condition 1 in the recursion for T_i . This property says that $\mathbf{g}(w_{\mathbf{g}'})$ is exactly the number of times that \mathbf{g}' is used. Thus, $\text{ind}(\mathbf{g})$ contains a quantitative information about the firing sequences which are allowed by \mathbf{g} . The condition says that (disregarding the real control by the sub-net N_i) the quantitative information is consistent with the expected control.

3. $\forall w \in C(N_i) \setminus (P_{T_i}^+ \cup P_{T_i}^-) \sum_{g \in \Gamma_t} \mathbf{g}(w) > 0$. This condition says that each witness appears in a period and, thus, the use of each interior transition and period is unlimited.
4. There are multisets $\exists \mathbf{m}_+, \mathbf{m}_- \in \mathbf{R}(N_i)$ with $\forall p \in P_{T_i}$

$$\mathbf{m}_+ |_{P_{T_i}^-} \in (\mathbf{c}_t + \Gamma_t^*) |_{P_{T_i}^-} \wedge ((\forall \mathbf{g} \in \Gamma_t \mathbf{g}(p^-) = 0) \rightarrow \mathbf{m}_+(p^+) > \mathbf{m}_+(p^-)) \wedge$$

$$\mathbf{m}_- |_{P_{T_i}^+} \in (\mathbf{c}_t + \Gamma_t^*) |_{P_{T_i}^+} \wedge ((\forall \mathbf{g} \in \Gamma_t \mathbf{g}(p^+) = 0) \rightarrow \mathbf{m}_-(p^-) > \mathbf{m}_-(p^+)).$$

This condition says that there is a firing sequence in the sub-net N_i quantitatively described by \mathbf{m}_+ . This firing sequence starts with a marking available by $\mathbf{c}_t + \Gamma_t^*$ and increases all those places which cannot be increased by Γ_t .

5. $\mathbf{c}_t |_{C(t)} \in \mathbf{R}(t)$. This condition says that transition t can fire without the use of one of its periods in Γ_t .

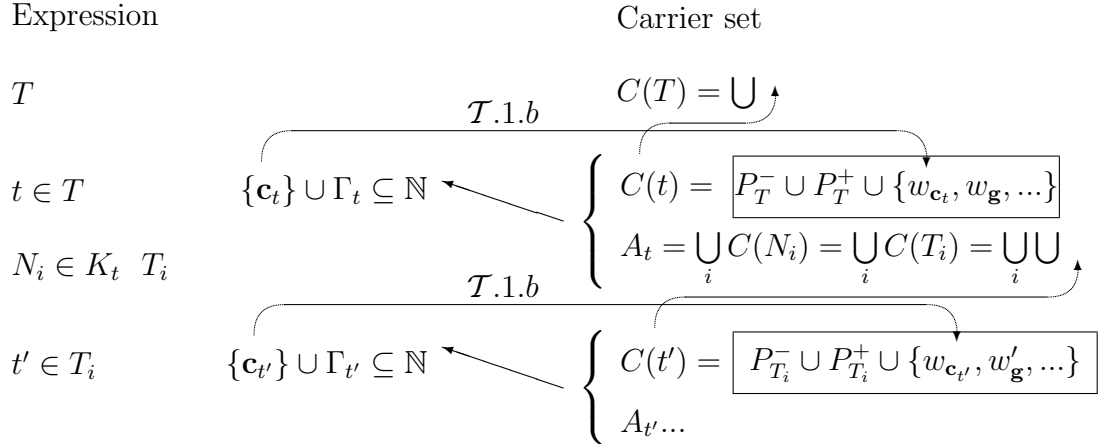


Figure 2: An overview over the expressions and their carrier sets.

In Section 4, we describe a decision algorithm which reduces expressions not having the property \mathcal{T} in every step. This leads to the following theorem:

Theorem 3.1 *For every expression T , we can effectively construct a T' with $\mathbf{R}(T) = \mathbf{R}(T')$ such that T' has property \mathcal{T} .*

Corollary 3.1 *The reachability problem for a Petri net with one inhibitor arc is decidable.*

Proof: According to Lemmata 2.3 and 3.1, we can construct an expression T where $\mathbf{R}(T) = \{\emptyset\}$ (and is not empty) if and only if there is a firing sequence $w \in T^*$ with $\mathbf{m}_0[w]\mathbf{m}_e$. Then, we construct T' according to Theorem 3.1. According to Condition 5 of property \mathcal{T} , $\mathbf{R}(T) = \mathbf{R}(T')$ is empty if and only if $T' = \emptyset$. ■

Remark: Alternatively, instead of using Lemma 2.3, we can construct T for a given net $N = (P, T, W, \{(p_1, \hat{t})\}, \mathbf{m}_0, \mathbf{m}_e)$ directly by connecting three sub-nets with a linear set in such a way that the first sub-net expresses firing sequences in $(T \setminus \{\hat{t}\})^*$ starting with \mathbf{m}_0 before the first firing of \hat{t} , the second contains only one transition which has a sub-net itself expressing firing sequences $\in (\hat{t}(T \setminus \{\hat{t}\})^*)^*$ ending with markings without tokens on p_1 and the third sub-net expresses firing sequences in $(T \setminus \{\hat{t}\})^*$ ending with \mathbf{m}_e .

3.2 The size of an expression

To prove the termination of the algorithm in Section 4, we have to define an ordering on a size S which is Noetherian and decreasing in every step of the algorithm:

A list (tuple, respectively) is smaller than another if the first i elements are equal and the $i + 1$ 'th element is smaller (or not existing). A multiset \mathbf{m} is smaller

than a multiset \mathbf{m}' if there is an e with $\mathbf{m}(e) < \mathbf{m}'(e)$ and $\mathbf{m}(e') = \mathbf{m}'(e')$ for all $e' > e$. (Thus multisets may as well be interpreted as a descending ordered list using lexicographic order.)

The smallest size is $S(\emptyset)$. Accordingly, if $T = \emptyset$ then T trivially has the property \mathcal{T} .

The size $S(T) = \sum_{t \in T} \{S(t) \mapsto 1\}$ is a multiset of all sizes $S(t)$ with $t \in T$. The size of t is $S(t) := (S(K_t), b_2, b_5 + |\Gamma_t|)$. Here, $b_i = 0$ if Condition $\mathcal{T}.i$ is fulfilled, and $b_i = 1$ otherwise. The size $S(K_t) = \sum_{N_i \in K_t} \{S(N_i) \mapsto 1\}$ of a set of nets is a multiset of the sizes $S(N_i)$ of the nets $N_i \in K_t$. The size of a net is

$$S(N_i) := (\mathbf{s}_m + \{|P_{T_i}| \mapsto 1\}, S(T_i), b_{1b}, |C(N_i)|)$$

with $\mathbf{s}_m := \max\{\mathbf{s} \mid \exists \mathbf{g}, f, b_2, b'_{1b}, e, \mathbf{s}' \mathbf{s}'((\mathbf{s}, \mathbf{g}, b'_{1b}, f)) > 0, S(T_i)((\mathbf{s}', b_2, e)) > 0\}$. In other words, the first component is a multiset in $\mathbb{N}^{\mathbb{N}}$ which is obtained by taking the maximal of such multisets of all first components in the size of a subnet of one of the transitions in T_i (respectively \emptyset if none exists) and adding the current number of places. The second component contains the recursion. The reason for this complicated construction comes from Section 4.4 where the recursion-depth increases but the size has to decrease. Furthermore, this causes $S(N_i)$ to be greater than the size of its occurring subnets. This is also necessary in parts where the algorithm works recursively since it follows that $S(K_{t'}) < S(K_t)$ for all t' contained one or more levels deeper in K_t .

Example (continued):

$$\begin{aligned} S(t_7) = S(t_8) &= (\emptyset, 0, 3), \quad S(T_1) = \{(\emptyset, 0, 3) \mapsto 2\}, \\ S(N_1) &= (\{3 \mapsto 1\}, \{(\emptyset, 0, 3) \mapsto 2\}, 1, 6), \quad S(t_2) = (\{S(N_1) \mapsto 1\}, 1, 4), \\ S(T_2) &= \{S(t'_2) \mapsto 1, (\emptyset, 0, 2) \mapsto 1\}, \quad S(N_2) = (\{3 \mapsto 1, 2 \mapsto 1\}, S(T_2), 1, 4). \end{aligned}$$

Lemma 3.2 *The ordering on S defined above is Noetherian*

Proof: As shown in [DM79], the set of descending ordered lists of elements of a Noetherian ordered set is again Noetherian. The first components of the quadruples $S(N)$ are descending lists of natural numbers and, thus, Noetherian.

Assume by contradiction that \mathbf{x} is the smallest first component such that there is an infinite descending sequence of quadruples

$$S(N) = (\mathbf{x}, \mathbf{y}_1, b'_1, n_1), (\mathbf{x}, \mathbf{y}_2, b'_2, n_2), \dots$$

In all quadruples appearing in all lists in all triples appearing in any \mathbf{y}_i , the first component must always be smaller than \mathbf{x} and, therefore, their order must be Noetherian. Thus, the lists which are the first components of the triples are also ordered Noetherian. Since the other components are natural numbers, the triples and the \mathbf{y}_i 's are also ordered Noetherian. Since the first component \mathbf{x} must remain constant, and the third and fourth components are natural numbers, we get a contradiction; thus, $S(T)$ is Noetherian. \blacksquare

3.3 Additional operators working on expressions

The following lemma is used to restrict the semilinear part in a transition t as it will be needed to establish the property $\mathcal{T}.2$

Lemma 3.3 *Let $t = L_t \hat{\circ}_{A_t} K_t$ be an expressions for a transition and L be (an expression for) a semi linear set. Then, we can construct an expression $T' = t|_L$ (with $\mathbf{R}(T') = (\mathbf{R}(L_t) \cap \mathbf{R}(L)) \hat{\circ}_{A_t} \mathbf{R}(K_t)$) where the occurring sizes $S(t')$ with $t' \in T'$ can increase relatively to $S(t)$ only in the last position in the triple.*

Proof: Using Presburger arithmetic [GS65],[ES69], we can calculate for every $t \in T$ the semi-linear set

$$L_t \cap L =: \bigcup_{j=1}^l L_j$$

resulting in finitely many linear sets L_j , and define $T' := t|_L := \{L_j \hat{\circ}_{A_t} K_t \mid j \leq l\}$. ■

An Example is given in Subsection 4.2 where $L_t \cap L$ consists of only one linear set and thus T' consists of only one transition.

The following Lemma 3.4 and Lemma 3.1 allow us to bring every expression into the normal form as nested Petri nets:

Lemma 3.4 *Let T and T' be expressions for sets of transitions, and Q be a relation. Then, we can construct an expression $T'' := T \circ_Q T'$ (with $\mathbf{R}(T'') = \mathbf{R}(T) \circ_Q \mathbf{R}(T')$) where the occurring sizes $S(t)$ can increase only in the last position in the triple and sum up in the first position.*

Proof: We may assume that $\bigcup_{t \in T} A_t$, $\bigcup_{t \in T'} A_t$ and $\pi_1(Q) \cup \pi_2(Q)$ are pairwise disjoint (otherwise replace elements by copies). We define

$$T'' := \left\{ L_j \hat{\circ}_{A_t \cup A_{t'}} (K_t + K_{t'}) \mid t \in T, t' \in T', j \leq r, L_t \circ_Q L_{t'} =: \bigcup_{j=1}^r L_j \right\}$$

using Presburger arithmetics (since $L_t \circ_Q L_{t'}$ is semilinear see Subsection 1.1.3).

It holds $\mathbf{R}(T) \circ_Q \mathbf{R}(T') = \bigcup_{t \in T, t' \in T'} \mathbf{R}(t) \circ_Q \mathbf{R}(t') =$

$$\begin{aligned} & \bigcup_{t \in T, t' \in T'} ((\mathbf{L}_t \hat{\circ}_{A_t} (\mathbf{R}(N_1) + \dots)) \circ_Q (\mathbf{L}_{t'} \hat{\circ}_{A_{t'}} (\mathbf{R}(N'_1) + \dots))) = \\ & \bigcup_{t \in T, t' \in T'} ((\mathbf{R}(N_1) + \dots) \hat{\circ}_{A_t^{-1}} \mathbf{L}_t \circ_Q \mathbf{L}_{t'} \hat{\circ}_{A_{t'}} (\mathbf{R}(N'_1) + \dots)) = \\ & \bigcup_{t \in T, t' \in T'} (\mathbf{L}_t \circ_Q \mathbf{L}_{t'}) \hat{\circ}_{A_t \cup A_{t'}} (\mathbf{R}(N_1) + \dots + \mathbf{R}(N'_1) + \dots) = \mathbf{R}(T'') \end{aligned}$$

since A_t , $A_{t'}$ and $\pi_1(Q) \cup \pi_2(Q)$ are pairwise disjoint. (see Subsection 1.1.2.) ■

Examples are given in Subsections 4.3 and 4.4.

From this we can conclude some decidability result for the first order formulas with PLUS and the *monotone transitive closure mTC* defined as follows:

Given a formula $\phi(x_1, \dots, x_k, x'_1, \dots, x'_k)$, then $\text{mTC}(\phi)$ denotes the smallest set $S \subset \mathbb{N}^{2k}$ containing all of the following:

- $(x_1, \dots, x_k, x_1, \dots, x_k)$ for $(x_1, \dots, x_k) \in \mathbb{N}^k$ (this stands for the identity),
- $(x_1, \dots, x_k, x'_1, \dots, x'_k)$ for $\phi(x_1, \dots, x_k, x'_1, \dots, x'_k)$,
- $(x_1, \dots, x_k, x''_1, \dots, x''_k)$ for $(x_1, \dots, x_k, x'_1, \dots, x'_k), (x'_1, \dots, x'_k, x''_1, \dots, x''_k) \in S$, and
- $(x_1 + x''_1, \dots, x_k + x''_k, x'_1 + x''_1, \dots, x'_k + x''_k)$ for a $(x_1, \dots, x_k, x'_1, \dots, x'_k) \in S$ and $(x''_1, \dots, x''_k) \in \mathbb{N}^k$.

Corollary 3.2 *The emptiness and satisfiability is decidable for formulas with an FO+PLUS-formula inside and \wedge, \vee, \exists and mTC operators outside.*

Proof: We can express linear sets by a t and, thus, semilinear sets by a T . Now, observe that the operators work on expressions of the form T as follows: We can express \wedge corresponding to \cap with \circ_Q (see Section 1.1.2) and apply Lemma 3.4. For \vee this follows simply from T being already a union. The existential quantifier is done by removing the element (thus, releasing the control from the outside) and the operator mTC is done by using Lemma 3.1. Then we construct T' according to Theorem 3.1. According to Condition 5 of property \mathcal{T} , $\mathbf{R}(T) = \mathbf{R}(T')$ is empty if and only if $T' = \emptyset$. ■

4 The main algorithm establishing property \mathcal{T}

The idea of the algorithm is to reduce T if one of the conditions is not fulfilled. For Condition 2, Presburger arithmetics is used to transfer the implicit quantitative restriction by the witness places to the explicit restriction of the transitions. Condition 3 ensures that all quantitative controls are unlimited. Condition 4 ensures that all places are unlimited. A covering graph construction deciding Condition 4 uses the algorithm recursively (like for Condition 1) for every step. Here, the current marking of a node is being included as a restriction to the semilinear set. Limited places are deleted at the cost of a larger structure. This larger structure, however, contains parts which are generated by restricting parts. This restriction might cause them to loose the property \mathcal{T} reached by a previous recursive step. However, as we will see because of their smaller size, the property \mathcal{T} can be established again and the whole algorithm will still terminate.

Proof:(of Theorem 3.1)

The expression T' in the Theorem is computed by the following algorithm where the details are explained in the subsections:

```

function reachedq( $T$ ):
begin
  repeat
     $i := 1$ 
    while  $i \leq 5$  and  $\forall t \in T, \forall N \in K_t$  Condition  $\mathcal{T}.i$  fulfilled
      do  $i := i + 1$  od
    if  $i = 6$  then return  $T$ 
    else  $T := T'$  for  $T'$  according to subsection 4.i fi
  until  $i = 6$ 
end reachedq

```

in each step $S(T)$ decreases ($S(\text{reachedq}(T)) < S(T)$ if $T \neq \text{reachedq}(T)$); due to Lemma 3.2 the algorithm terminates. \blacksquare

The following table shows how the size $S(t)$ can change during the steps of Chapter 4:

| | $S(t)$ | | | | | | | | | |
|-----|---------------------------------|--------------------------|------------|-----------------------|--|--------------|--------------|--------------|--------------|--------------------|
| | $s_m + \{ P_{T_i} \mapsto 1\}$ | $S(N_i)$ | | | | | b_{1b} | $ C(N_i) $ | b_2 | $b_5 + \Gamma_t $ |
| | | $S(t')$ for $t' \in T_i$ | | | | | | | | |
| | | $S(K_{t'})$ | b_2 | $b_5 + \Gamma_{t'} $ | | | | | | |
| 4.1 | - | - | - | - | | \downarrow | \uparrow | \uparrow | - | |
| 4.2 | - | - | - | - | | - | - | \downarrow | \uparrow | |
| 4.3 | - | - | - | - | | - | \downarrow | \uparrow | \uparrow | |
| 4.4 | \downarrow | \uparrow | \uparrow | \uparrow | | \uparrow | \uparrow | \uparrow | \uparrow | |
| 4.5 | - | - | - | - | | - | - | - | \downarrow | |

4.1 Condition 1 Recursion and introducing witnesses

Let Condition 1 be not fulfilled by T_i ; let $T'_i := \text{reachedq}(T_i)$, which terminates by induction since $S(T_i) < S(T)$.

For all $t_j \in T'_i$ let G_j be the set of all $\mathbf{g} \in \{\mathbf{c}_{t_j}\} \cup \Gamma_{t_j}$ not having a witness. Add witnesses $\{w_{\mathbf{g}'} \mid \mathbf{g} \in G_j\}$ to $C(t'_j) := C(t_j) \cup \{w_{\mathbf{g}'} \mid \mathbf{g} \in G_j\}$ by replacing each \mathbf{g} in G_j by $\mathbf{g}' := \mathbf{g} + \{w_{\mathbf{g}'} \mapsto 1\}$ in

$$T''_i := \left\{ t'_j \mid \begin{array}{l} t_j \in T'_i, A_{t'_j} = A_{t_j}, K_{t'_j} = K_{t_j}, \\ \Gamma_{t'_j} = \Gamma_{t_j} \setminus G_j \cup \{\mathbf{g}' \mid \mathbf{g} \in G_j \setminus \{\mathbf{c}_t\} \text{ and } \mathbf{c}_{t'_j} := \mathbf{c}_{t_j} \text{ if } \mathbf{c}_{t_j} \notin G_j \} \end{array} \right\}.$$

Now, we set $t' := L_{t'} \hat{\circ}_{A_{t'}} K_{t'}$ with $K_{t'} = K_t \setminus \{N_i\} \cup \{N''_i\}$, $N''_i = *_{P_{T''_i}}(T''_i)$, $A_{t'} = A_t \cup C(N''_i)$ and $\Gamma_{t'} := \Gamma_t \cup \{\{w \mapsto 1\} \mid w \in C(N''_i) \setminus C(N_i)\}$, and let $T' := T \setminus \{t\} \cup \{t'\}$. Since $\mathbf{R}(T_i) = \mathbf{R}(T''_i) |_{C(T_i)}$, we have $\mathbf{R}(N_i) = \mathbf{R}(N''_i) |_{C(T_i)}$; thus, $\mathbf{R}(t) = \mathbf{R}(t')$; thus, $\mathbf{R}(T) = \mathbf{R}(T')$.

Since $S(t_j) = S(t'_j)$ for all $t_j \in T'_i$, the size $S(T''_i) = S(T'_i)$ remains unchanged. The only increase was $|C(N''_i)| \geq |C(N_i)|$ but we have either $S(T''_i) = S(T'_i) <$

$S(T_i)$, or in case $T'_i = T_i$, we have now $b_{1b} = 0$. From that follows that $S(N''_i) < S(N_i)$; thus, $S(t') < S(t)$ and $S(T') < S(T)$.

Example (continued):

Since the expression T_1 does not fulfill Condition 1, we add the two witnesses $w_{c_{t'_7}}$ and $w_{c_{t'_8}}$. For simplicity, we omit the witnesses for the periods for \mathbf{Id}_P in the elementary transitions. So we replace t_7 and t_8 by $t'_7 = \{\hat{p}_2^- \mapsto 1, \hat{p}_1^+ \mapsto 3, w_{c_{t'_7}} \mapsto 1\} + \mathbf{Id}_{\hat{p}_1, \hat{p}_2, \hat{p}_3}$ and $t'_8 = \{\hat{p}_1^- \mapsto 2, \hat{p}_3^+ \mapsto 1, w_{c_{t'_8}} \mapsto 1\} + \mathbf{Id}_{\hat{p}_1, \hat{p}_2, \hat{p}_3}$.

This yields the expressions $T''_1 = t'_7 \cup t'_8$ and $N''_1 = \star_{\{\hat{p}_1, \hat{p}_2, \hat{p}_3\}}(T''_1)$. On the next level, we get $t'_2 = (\emptyset + \{\{p_2^- \mapsto 1, \hat{p}_2^- \mapsto 1\}, \{p_3^- \mapsto 1, \hat{p}_3^- \mapsto 1\}, \{p_2^+ \mapsto 1, \hat{p}_2^+ \mapsto 1\}, \{p_3^+ \mapsto 1, \hat{p}_3^+ \mapsto 1\}, \{w_{c_{t'_7}} \mapsto 1\}, \{w_{c_{t'_8}} \mapsto 1\}\}^*), \hat{\circ}_{\{\hat{p}_2^-, \hat{p}_3^-, \hat{p}_2^+, \hat{p}_3^+\}} N''_1$ for the generalized transition and $T'_2 = t'_2 \cup \hat{t}$.

The new sizes are now $S(t'_7) = S(t'_8) = (\emptyset, 0, 3) = S(t_7)$,

$S(T''_1) = \{(\emptyset, 0, 3) \mapsto 2\} = S(T_1)$,

$S(N''_1) = (\{3 \mapsto 1\}, \{(\emptyset, 0, 3) \mapsto 2\}, 0, 8) < S(N_1)$,

$S(t'_2) = (\{S(N''_1) \mapsto 1\}, 1, 6) < S(t_2)$,

$S(T'_2) = \{S(t'_2) \mapsto 1, (\emptyset, 0, 2) \mapsto 1\} < S(T_2)$.

4.2 Condition 2 Quantitative consistency

Let Condition 2 be not fulfilled by T_i . The set $\mathbf{L} :=$

$$\{\mathbf{g} \in \mathbb{N}^{C_L} \mid \forall p \in \bigcup_{N_i \in K_t} P_{T_i} \mathbf{g}(p^-) - \mathit{ind}(\mathbf{g})(p^-) = \mathbf{g}(p^+) - \mathit{ind}(\mathbf{g})(p^+) \}$$

on the carrier set $C_L = C(t) \cup \bigcup_{N_i \in K_t} C(N_i)$ is a Presburger set. Since $\mathbf{R}(t) \subseteq \mathbf{L}|_{C(t)}$

follows from the definition of $\mathbf{R}(t)$ and the function ind , we can set $T' := T \setminus \{t\} \cup t|_L$ using Lemma 3.3. In other words, we have cut something away which could not have been in $\mathbf{R}(T)$ anyway.

Since b_2 is now 0 for each $t_j \in t|_L$ and $S(K_{t_j})$ remains the same as $S(K_t)$, according to Lemma 3.3, it holds $S(T') < S(T)$.

Example (continued):

We see that t'_2 does not fulfill Condition 2 when we look at the resulting equation

$$\mathbf{g}(\hat{p}^-) - \mathbf{g}(w_{c_{t'_7}}) \mathbf{c}_{t'_7}(\hat{p}^-) - \mathbf{g}(w_{c_{t'_8}}) \mathbf{c}_{t'_8}(\hat{p}^-) = \mathbf{g}(\hat{p}^+) - \mathbf{g}(w_{c_{t'_7}}) \mathbf{c}_{t'_7}(\hat{p}^+) - \mathbf{g}(w_{c_{t'_8}}) \mathbf{c}_{t'_8}(\hat{p}^+)$$

for all $p \in P$ characterizing L . This is equivalent to the following three equations: $2\mathbf{g}(w_{c_{t'_8}}) = 3\mathbf{g}(w_{c_{t'_7}})$, $\mathbf{g}(\hat{p}_2^-) - \mathbf{g}(w_{c_{t'_7}}) = \mathbf{g}(\hat{p}_2^+)$, $\mathbf{g}(\hat{p}_3^-) = \mathbf{g}(\hat{p}_3^+) - \mathbf{g}(w_{c_{t'_8}})$. Their solutions are described by the linear set $L_{t''_2} = L_{t'_2} \cap L =$

$$\emptyset + \left\{ \left[\begin{array}{cccc} p_2^- & \hat{p}_2^- & p_2^+ & \hat{p}_2^+ \\ 1 & 1 & 1 & 1 \end{array} \right], \left[\begin{array}{cccc} p_3^- & \hat{p}_3^- & p_3^+ & \hat{p}_3^+ \\ 1 & 1 & 1 & 1 \end{array} \right], \left[\begin{array}{cccc} w_{c_{t'_7}} & w_{c_{t'_8}} & p_2^- & \hat{p}_2^- & p_3^- & \hat{p}_3^- \\ 2 & 3 & 2 & 2 & 3 & 3 \end{array} \right] \right\}^*$$

and yield $t''_2 = t'_2|_L = L_{t''_2} \hat{\circ}_{\{\hat{p}_2^-, \hat{p}_2^+, \hat{p}_3^-, \hat{p}_3^+\}} N''_1$ with $S(t''_2) = (\{S(N''_1) \mapsto 1\}, 0, 3) < S(t'_2)$. Since $T''_2 = t''_2 \cup \hat{t}$ fulfills the remaining properties, we can continue one level higher.

Adding the witnesses leads to $L_{t_2''} =$

$$\emptyset + \left\{ \left[\begin{array}{cccccc} p_2^- & \hat{p}_2^- & p_2^+ & \hat{p}_2^+ & w_1 & \\ 1 & 1 & 1 & 1 & 1 & \end{array} \right], \left[\begin{array}{cccccc} p_3^- & \hat{p}_3^- & p_3^+ & \hat{p}_3^+ & w_2 & \\ 1 & 1 & 1 & 1 & 1 & \end{array} \right], \left[\begin{array}{cccccc} w_{c_{t_7}} & w_{c_{t_8}} & p_2^- & \hat{p}_2^- & p_3^- & \hat{p}_3^- \\ 2 & 3 & 2 & 2 & 3 & 3 \end{array} \right] \right\}^*$$

(we omit the witness for \emptyset .) with $S(t_2''') = S(t_2'') = (\{S(N_1'') \mapsto 1\}, 0, 3)$.

Defining $T_2''' = t_2''' \cup \hat{t}'$ with $S(T_2''') = S(T_2'')$ and $N_2''' = \star_{\{\hat{p}_1, \hat{p}_2, \hat{p}_3\}}(T_2''')$ with $S(N_2''') = (\{3 \mapsto 1, 2 \mapsto 1\}, S(T_2'''), 0, 8) < S(N_2'') = (\{3 \mapsto 1, 2 \mapsto 1\}, S(T_2''), 1, 4)$ we get

$$t_3'' = \left(\left[\begin{array}{cccc} p_2^- & p_3^- & p_2^+ & p_3^+ \\ 4 & 2 & 4 & 3 \end{array} \right] + \left\{ \left[\begin{array}{c} w_1 \\ 1 \end{array} \right], \left[\begin{array}{c} w_2 \\ 1 \end{array} \right], \left[\begin{array}{c} w_3 \\ 1 \end{array} \right], \left[\begin{array}{c} w_{c_{t_7}} \\ 1 \end{array} \right] \right\}^* \right) \circ_{\{p_2^-, p_2^+, p_3^-, p_3^+\}} N_2''.$$

Establishing Condition 2 leads to

$$t_3'' = \left(\left[\begin{array}{cccccc} p_2^- & p_3^- & p_2^+ & p_3^+ & w_3 & w_{c_{t_7}} \\ 4 & 2 & 4 & 3 & 5 & 2 \end{array} \right] + \left\{ \left[\begin{array}{c} w_1 \\ 1 \end{array} \right], \left[\begin{array}{c} w_2 \\ 1 \end{array} \right] \right\}^* \right) \circ_{\{p_2^-, p_2^+, p_3^-, p_3^+\}} N_2''.$$

4.3 Condition 3 Elimination of witnesses

Let Condition 3 be not fulfilled by witness $w \in C(N_i) \setminus (P_{T_i}^+ \cup P_{T_i}^-)$. This means that we can replace N_i by some expression \hat{T} with $\mathbf{R}(\hat{T}) = \mathbf{R}(N_i) \hat{\circ}_{\{w\}} \mathbf{c}_t|_w$ since for all $\mathbf{m} \in \mathbf{L}_t$, we have $\mathbf{m}(w) = \mathbf{c}_t(w)$. Then, we can set

$$T' := T \setminus \{t\} \cup (L_t|_{\overline{\{w\}}} \hat{\circ}_{A_t \setminus C(N_i)} (K_t \setminus \{N_i\})) \hat{\circ}_{C(N_i) \setminus \{w\}} \hat{T}$$

which replaces the transition $t = L_t \circ_{A_t} K_t$ by all those sets of transitions which result from using Lemma 3.4 (because \hat{T} is not a net). This means that N_i is removed and the equivalent \hat{T} is plugged in at the same range; thus, $\mathbf{R}(t) = \mathbf{R}(L_t)|_{\overline{\{w\}}} \hat{\circ}_{A_t \setminus \{w\}} (\mathbf{R}(K_t \setminus \{N_i\}) + \mathbf{R}(\hat{T}))$.

To create $\hat{T} = \bigcup_{\gamma} T_{\gamma}$, we consider every possible combination γ (including the order

of the summands) of $\mathbf{c}_t(w) = \sum_{m=1}^{l_{\gamma}} \mathbf{g}_m(w)$ with $\mathbf{g}_m \in \mathbf{c}_{t_m} + \{\mathbf{g} \in \Gamma_{t_m} \mid \mathbf{g}(w) > 0\}^*$, $\mathbf{g}_m(w) > 0$ and $t_m \in T_i$ and build t'_m with $L_{t'_m} = \mathbf{g}_m|_{\overline{\{w\}}} + \{\mathbf{g} \in \Gamma_{t_m} \mid \mathbf{g}(w) = 0\}^*$ and $K_{t'_m} := K_{t_m}$. The expressions t'_m describe the parts in which w was used. In $N'_i = \star_{P_{T'_i}}(T'_i)$ with $T'_i :=$

$$\{t''' \mid t'' \in T_i, \mathbf{c}_{t''} = \mathbf{c}_{t''}, K_{t''} = K_{t''}, \mathbf{c}_{t''}(w) = 0, \Gamma_{t''} = \{\mathbf{g} \in \Gamma_{t''} \mid \mathbf{g}(w) = 0\}\},$$

we filter out everything which affects w ; thus, $C(N'_i) = C(T'_i) = C(T_i) \setminus \{w\}$ and $\mathbf{R}(N'_i) = \{\mathbf{m} \in \mathbf{R}(N_i) \mid \mathbf{m}(w) = 0\}$. Then, using Lemma 3.1, we construct $t(N'_i)$ which has now the property $\mathbf{R}(t(N'_i)) = \{\mathbf{m} \in \mathbf{R}(N_i) \mid \mathbf{m}(w) = 0\}$. Now, we define

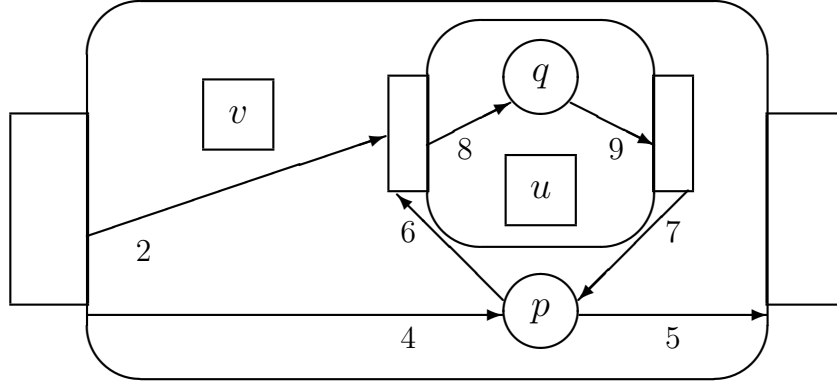
$$T_{\gamma} := t(N'_i) \circ_{P_{T'_i}} t'_1 \circ_{P_{T'_i}} t(N'_i) \circ_{P_{T'_i}} t'_2 \circ_{P_{T'_i}} \dots \circ_{P_{T'_i}} t'_{l_{\gamma}} \circ_{P_{T'_i}} t(N'_i)$$

again using Lemma 3.4.

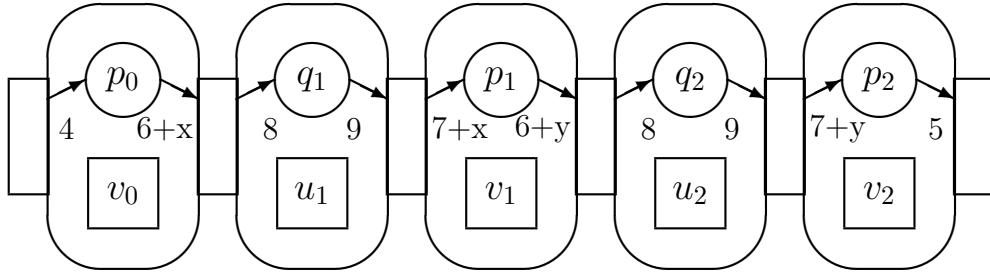
It holds $S(K_{t'}) < S(K_t)$ for every new t' in T' because of $S(N'_i) < S(N_i)$. This in turn follows from $|C(T'_i)| = |C(T_i)| - 1$ and $S(N) < S(N_i)$ for all $N \in K_{t_m}$, and $m \leq l_\gamma$ for all γ .

It holds $S(T') < S(T)$ since $S(t') < S(t)$ for every t' .

Example: Consider t with $\mathbf{c}_t = \begin{bmatrix} w & p^- & p^+ \\ 2 & 4 & 5 \end{bmatrix}$, $\forall \mathbf{g} \in \Gamma_t \mathbf{g}(w) = 0$, $K_t = \{\star_{\{p\}}(v \cup t_j)\}$, and $\mathbf{c}_{t_j} = \begin{bmatrix} w & p^- & p^+ & q^- & q^+ \\ 1 & 6 & 7 & 8 & 9 \end{bmatrix}$, $K_{t_j} = \{\star_{\{q\}}(u)\}$.



Then t' is defined such that $\mathbf{c}_{t'} = \begin{bmatrix} p_0^- & p_0^+ & q_1^- & q_1^+ & p_1^- & p_1^+ & q_2^- & q_2^+ & p_2^- & p_2^+ \\ 4 & 6 & 8 & 9 & 7 & 6 & 8 & 9 & 7 & 5 \end{bmatrix}$, furthermore, $\begin{bmatrix} p_1^- & p_0^+ \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} p_2^- & p_1^+ \\ 1 & 1 \end{bmatrix} \in \Gamma_{t'}$ and $K_{t'} = \{\star_{\{p_0\}}(v_0), \star_{\{q_1\}}(u_1), \star_{\{p_1\}}(v_1), \star_{\{q_2\}}(u_2), \star_{\{p_2\}}(v_2)\}$, where p_i, q_i, v_i and u_i are replacements caused by disjointness condition in Lemma 3.4.



The variables x and y illustrate the effect of the periods in $\Gamma_{t'}$ which originate from the (omitted) periods of t_j .

4.4 Condition 4 Elimination of bounded places

Condition 4 is decidable by two *covering graph* constructions for every i working as follows: Every node in the covering graph $CG_{(i,+)}$ ($CG_{(i,-)}$, respectively) has a marking from $(\mathbb{N} \cup \{\omega\})^{P_{T_i}^-}$ ($(\mathbb{N} \cup \{\omega\})^{P_{T_i}^+}$, respectively). The root of the covering graph $CG_{(i,+)}$ has the marking $\mathbf{c}_t |_{P_{T_i}^-} + \omega^{\{p^- \mid \exists \mathbf{g} \in \Gamma \mathbf{g}(p^-) > 0\}}$.

For a node in $CG_{(i,+)}$ marked with \mathbf{m} , we construct T'_i with $\mathbf{R}(T'_i) = \{\mathbf{g} \in \mathbf{R}(T_i) \mid \mathbf{g}|_{P_{T_i}^-} \leq \mathbf{m}\}$ using Lemma 3.3 as $T'_i := \{t' \mid \{\mathbf{g} \in \mathbf{L}_{t'} \mid \mathbf{g}|_{P_{T_i}^-} \leq \mathbf{m}\} \mid t' \in T_i\}$. This restricts the allowed multisets to those which are possible starting with the limited marking \mathbf{m} . All $K_{t'}$ with $t' \in T'_i$ appear in the subnet N_i in t (unchanged by Lemma 3.3). For all $N' \in K_{t'}$, we have $S(N') < S(N_i)$ since the first component in $S(N_i)$ is $\{|P_{T_i}| \mapsto 1\}$ plus the maximum of everything one level deeper. Therefore, we have $S(K_{t'}) < S(K_t)$ for all $t' \in T'_i$ and, thus, $S(T'_i) < S(T)$. This allows us to compute $T''_i := \text{reacheq}(T'_i)$ recursively.

For every $t'' \in T''_i$, (since we know from Condition $\mathcal{T}.5$ that $\mathbf{c}_{t''}$ alone can fire), we add a new node

$$\mathbf{m}' := \mathbf{m} - \mathbf{c}_{t''}|_{P_{T_i}^-} + \{p^- \mapsto (\mathbf{c}_{t''}(p^+) + \omega \sum_{\mathbf{g} \in \Gamma_{t''}} \mathbf{g}(p^+)) \mid p \in P_{T_i}\}$$

to the covering graph $CG_{(i,+)}$. According to Corollary 4.1, there is no limit for the number of appearances of the multi-sets in $\Gamma_{t''}$ in firing sequences. This allows us to label those places p^- with ω where $\mathbf{g}(p^+) > 0$ for a $\mathbf{g} \in \Gamma_{t''}$.

If $\mathbf{m}' > \mathbf{m}''$ for an \mathbf{m}'' on the path from the root to \mathbf{m} , then we set $\mathbf{m}' := \mathbf{m}' + \omega(\mathbf{m}' - \mathbf{m}'')$. This is because we can lift the marking of those places p^- with $(\mathbf{m}' - \mathbf{m}'')(p^-) > 0$ by repeating the firing sequence corresponding to the path from \mathbf{m}'' to \mathbf{m}' arbitrarily many times.

If $\mathbf{m}' \leq \mathbf{m}'' \in \text{Path}(\mathbf{m}')$, then we need not calculate the successors of \mathbf{m}' since we already had better chances at \mathbf{m}'' .

According to [Dic13], there are only finite sets of incomparable multi-sets over a finite set $P_{T_i}^-$. It, therefore, follows that every path must terminate.

If for all i a node marked with $\omega^{P_{T_i}^-}$ is in $CG_{(i,+)}$ and, analogously, a node marked with $\omega^{P_{T_i}^+}$ is in $CG_{(i,-)}$, then the Condition 4 is fulfilled. Otherwise, we can calculate without loss of generality

$$k := \min_{\sigma \in \{+,-\}} \max_{\text{path} \subseteq CG_{(i,\sigma)}} \min_{p \in P_{T_i}} \max_{\mathbf{m} \in \text{path}} \mathbf{m}(p^\sigma)$$

This means that in every path in $CG_{(i,+)}$ or $CG_{(i,-)}$, there is a place p such that on this path there are never more than k tokens on p^- or p^+ respectively.

Now, we can replace in $T' := T \setminus \{t\} \cup \bigcup_{p \in P_{T_i}} U(p)$ the transition t by all those sets

of transitions $U(p)$, described in the following sub section, which are generated by restricting t in such a way that, in the subnet N_i , there can never be more than k tokens on p .

In order to show that $S(T') < S(T)$ we have to show that each $S(t') < S(t)$ for every t' in every $U(p)$.

4.4.1 Elimination of places

As in the construction of a regular expression from a finite automaton having the states $0, \dots, k$, we define for all $l, j, h \leq k$ an expression $T_{j,h}^{l-1}$ describing correspond-

ing firing sequences with the following property: They start with a marking \mathbf{m}_0 with $\mathbf{m}_0(p) = j$, end with a marking \mathbf{m}_1 with $\mathbf{m}_1(p) = h$, and meanwhile the number tokens on p is always less than l . This allows us to remove the place p since its information is no longer necessary. Therefore, we have $P_{T_{j,h}^{l-1}} = P_{T_i}^l := P_{T_i} \setminus \{p\}$. For an inductive definition, we start with the case of an immediate success where there is no 'meanwhile': This means

$$T_{j,h}^{-1} := T_i \hat{\circ}_{\{p^-, p^+\}} \{\{p^- \mapsto j, p^+ \mapsto h\}\}$$

is constructed using Lemma 3.4. (We can write $\{\{p^- \mapsto j, p^+ \mapsto h\}\}$ as $\{t_{j,h}\}$ with $\mathbf{c}_{t_{j,h}} = \{p^- \mapsto j, p^+ \mapsto h\}$ and $\Gamma_{t_{j,h}} = K_{t_{j,h}} = \emptyset$.) Recursively, we define

$$T_{l,l}^l := \{t(N_{l,l}^{l-1})\} := \{t(\star_{P_{T_i}^l}(T_{l,l}^{l-1}))\}$$

using Lemma 3.1. Then with Lemma 3.4, we construct

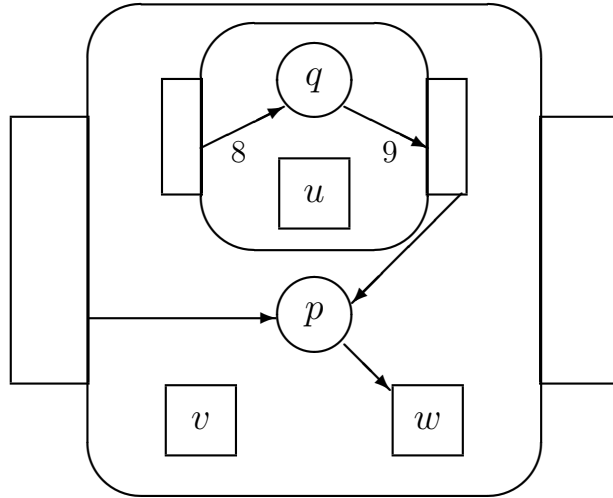
$$\begin{aligned} T_{l,h}^l &:= T_{l,l}^l \circ_{P_{T_i}^l} T_{l,h}^{l-1} \text{ for } h \neq l, \\ T_{j,l}^l &:= T_{j,l}^{l-1} \circ_{P_{T_i}^l} T_{l,l}^l \text{ for } j \neq l, \text{ and} \\ T_{j,h}^l &:= T_{j,l}^{l-1} \circ_{P_{T_i}^l} T_{l,l}^l \circ_{P_{T_i}^l} T_{l,h}^{l-1} \cup T_{j,h}^{l-1} \text{ for } h \neq l \wedge j \neq l. \end{aligned}$$

Now we define

$$U(p) = (L_t |_{\overline{\{p^-, p^+\}}} \hat{\circ}_{A_t \setminus C(N_i)} (K_t \setminus \{N_i\})) \hat{\circ}_{C(N_i) \setminus \{p^-, p^+\}} T_{\mathbf{c}_t(p^-), \mathbf{c}_t(p^+)}^k$$

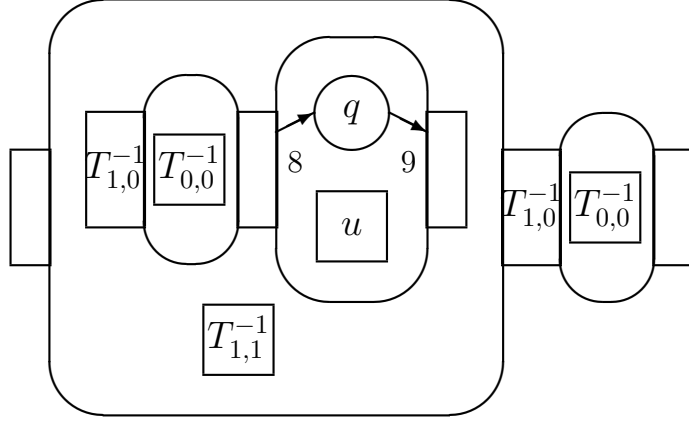
using Lemma 3.4. We have $S(N') < S(N_i)$ for every $N' \in K_{t'}$ with $t' \in T_{\mathbf{c}_t(p^-), \mathbf{c}_t(p^+)}^k$ because for the corresponding first components s' and s_i of the 4-tuples, we have $\mathbf{s}'(|P_{T_i}^l|) = \mathbf{s}_i(|P_{T_i}^l|) + k + 1$ but $\mathbf{s}'(|P_{T_i}^l|) = \mathbf{s}_i(|P_{T_i}^l|) - 1$ (It holds $|P_{T_i}^l| = |P_{T_i}^l| - 1$). Thus, $S(t'') < S(t)$ for every $t'' \in U(p)$.

Example: Let $t = (\mathbf{c} + \Gamma^*) \hat{\circ}_{C(N_i)} \star_{\{p\} \cup P}(N_i)$ with $\mathbf{c}(p^-) = 1$, $N_i = v \cup w \cup t_j$, $C(N_i) = \{p^-, p^+\} \cup P^- \cup P^+$ and $t_j = (\mathbf{c}_j + \Gamma_j^*) \hat{\circ}_{\{q^-, q^+\} \cup Q^- \cup Q^+} \star_{\{q\} \cup Q}(u)$ with $\mathbf{c}_j(p^+) = 1$, $\mathbf{c}_j(q^-) = 8$ and $\mathbf{c}_j(q^+) = 9$ look like



and $k = 1$. Furthermore, we assume no other occurrence of p in any other constant or period. This means that the firing sequences are restricted to the regular expression $((wv^*t_j)+v)^*wv^*$. This corresponds to $T_{0,0}^{-1}$ and $T_{1,1}^{-1}$ to consist only of a copy of v , $T_{1,0}^{-1}$ only of a copy of w and $T_{0,1}^{-1}$ only of a copy of t_j .

We get $T_{0,0}^0 = t(\star_P(T_{0,0}^{-1}))$, $T_{1,1}^0 = T_{1,0}^{-1} \circ_P T_{0,0}^0 \circ_P T_{0,1}^{-1} \cup T_{1,1}^{-1}$; in the end every new transition t' in $(\mathbf{c} + \Gamma^*) \big|_{\{p^-, p^+\}} \hat{\circ}_{C(N_i) \setminus \{p^-, p^+\}} T_{1,0}^1$ with $T_{1,0}^1 = T_{1,1}^0 \circ_P T_{1,0}^0 = t(\star_P(T_{1,1}^0)) \circ_P T_{1,0}^{-1} \circ_P T_{0,0}^0$ now looks like



In Section 5, we will show that we can build up firing sequences which compensate the 'odd' firing sequences from condition 4, from the constant, and from the 'odd' indirect firing sequences in order to find a \mathbf{c}_t fulfilling condition 5:

Lemma 4.1 *If the conditions 1 - 4 hold for t , then it holds*

$$\forall \mathbf{f} \in \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g} + \Gamma_t^* \forall \mathbf{e} \in (\Gamma_t \cup -\Gamma_t)^* \exists k \geq 2 \{(\mathbf{c}_t + k\mathbf{f})|_{C(t)}, (\mathbf{c}_t + k\mathbf{f} + \mathbf{e})|_{C(t)}\} \subseteq \mathbf{R}(t)$$

The proof is in the following section. From this immediately follows:

Corollary 4.1 *If the conditions 1 - 4 hold for t , then it holds*

$$\forall \mathbf{f} \in \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g} + \Gamma_t^* \exists k \geq 2 (\mathbf{c}_t + k\mathbf{f})|_{C(t)} \in \mathbf{R}(t)$$

4.5 Condition 5 Making the constant firing

If Condition 5 is not fulfilled for t then, according to Corollary 4.1, for $\mathbf{f} = \sum_{\mathbf{g} \in \Gamma} \mathbf{g}$, there exists a (smallest) k such that $(\mathbf{c}_t + k\mathbf{f})|_{C(t)} \in \mathbf{R}(t)$. So we decompose L_t such that $\mathbf{R}(L_t) = \mathbf{R}(L_t + k\mathbf{f}) \cup \bigcup_{\mathbf{g} \in \Gamma} \bigcup_{j \leq k} \mathbf{R}(\mathbf{c}_t + j\mathbf{g} + (\Gamma_t \setminus \{\mathbf{g}\})^*)$. Set

$$T' := T \setminus \{t\} \cup \{t' \mid K_{t'} = K_t, \Gamma_{t'} = \Gamma_t \wedge \mathbf{c}_{t'} = \mathbf{c}_t + k\mathbf{f}\} \\ \cup \{t' \mid \exists j \leq k, \mathbf{g} \in \Gamma \Gamma_{t'} = \Gamma_t \setminus \{\mathbf{g}\} \wedge \mathbf{c}_{t'} = \mathbf{c}_t + j\mathbf{g}\}.$$

Since Conditions 1 and 2 are not affected, b_2 and $S(K_t)$ do not change. The size $S(t')$ is smaller than $S(t)$ since b_5 is now zero respectively $|\Gamma \setminus \{\mathbf{g}\}| < |\Gamma|$; thus, it holds $S(T') < S(T)$.

5 Building up compensating firing sequences

Proof:(of Lemma 4.1) Given $\mathbf{f} \in \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g} + \Gamma_t^*$ and $\mathbf{e} \in (\Gamma_t \cup -\Gamma_t)^*$, we have to find a $k \geq 2$ such that $\{(\mathbf{c}_t + k\mathbf{f})|_{C(t)}, (\mathbf{c}_t + k\mathbf{f} + \mathbf{e})|_{C(t)}\} \subseteq \mathbf{R}(t)$.

For an elementary transition t with $K_t = \emptyset$, we have $\mathbf{R}(t) = \mathbf{c}_t + \Gamma_t^*$ and the statement is easily fulfilled by choosing a sufficiently large k compensating negative components in e .

Induction step: For every $N_i \in K_t$, we consider $\mathbf{m}_+, \mathbf{m}_- \in \mathbf{R}(N_i)$ according to Condition $\mathcal{T}.4$ and define $\mathbf{d} := -\mathbf{m}_+ - \mathbf{m}_-$. For every $N_i \in K_t$ and for every $t_j \in T_i$, let

$$\mathbf{f}_j := \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{f}(w_{\mathbf{g}})\mathbf{g}, \quad \mathbf{e}_j := \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{e}(w_{\mathbf{g}})\mathbf{g}, \quad \mathbf{h}_j := \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{c}_t(w_{\mathbf{g}})\mathbf{g}, \quad \mathbf{d}_j := \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{d}(w_{\mathbf{g}})\mathbf{g}.$$

Since, according to Condition $\mathcal{T}.3$, $\mathbf{f}(w_{\mathbf{g}}) > 0$ for every $\mathbf{g} \in \Gamma_{t_j}$, we have $\mathbf{f}_j \in \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{g} + \Gamma_{t_j}^*$. This means \mathbf{f}_j fulfills the condition for \mathbf{f} one level deeper.

By Condition $\mathcal{T}.1$ and by applying the lemma by induction, for sub-transitions t_j of t three times for \mathbf{e} as $\mathbf{e}_j, \mathbf{h}_j$ or \mathbf{d}_j and for \mathbf{f} as \mathbf{f}_j , there exist $k_j, k'_j, k''_j \geq 2$ with

$$\begin{aligned} &(\mathbf{c}_{t_j} + k_j\mathbf{f}_j)|_{C(t_j)}, (\mathbf{c}_{t_j} + k_j\mathbf{f}_j + \mathbf{e}_j)|_{C(t_j)}, \\ &(\mathbf{c}_{t_j} + k'_j\mathbf{f}_j)|_{C(t_j)}, (\mathbf{c}_{t_j} + k'_j\mathbf{f}_j + \mathbf{h}_j)|_{C(t_j)}, \\ &(\mathbf{c}_{t_j} + k''_j\mathbf{f}_j)|_{C(t_j)}, (\mathbf{c}_{t_j} + k''_j\mathbf{f}_j + \mathbf{d}_j)|_{C(t_j)} \in \mathbf{R}(t_j). \end{aligned} \quad (1)$$

From Condition $\mathcal{T}.4$, it follows that there exists a sufficiently large $h \geq 1$ with $h\mathbf{f} + \mathbf{e} \in \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g} + \Gamma_t^*$, such that for all i and j

$$\begin{aligned} \forall p \in P_{T_i} \quad \Delta_+(p) &:= h\mathbf{f}(p^-) - \mathbf{m}_+(p^-) + \mathbf{m}_+(p^+) \geq 1 \wedge \\ \Delta_-(p) &:= h\mathbf{f}(p^+) + \mathbf{m}_-(p^-) - \mathbf{m}_-(p^+) \geq 1, \end{aligned} \quad (2)$$

$h = n_j k_j = n'_j k'_j = n''_j k''_j$ for some n_j, n'_j, n''_j ,

$$\begin{aligned} l_j &:= n_j(k_j\mathbf{f}(w_{\mathbf{c}_{t_j}}) - 1) + \mathbf{e}(w_{\mathbf{c}_{t_j}}) > 0, \\ n'_j(k'_j\mathbf{f}(w_{\mathbf{c}_{t_j}}) - 1) + \mathbf{c}_t(w_{\mathbf{c}_{t_j}}) &> 0 \text{ and} \\ n''_j(k''_j\mathbf{f}(w_{\mathbf{c}_{t_j}}) - 1) + \mathbf{d}(w_{\mathbf{c}_{t_j}}) &> 0 \end{aligned} \quad (3)$$

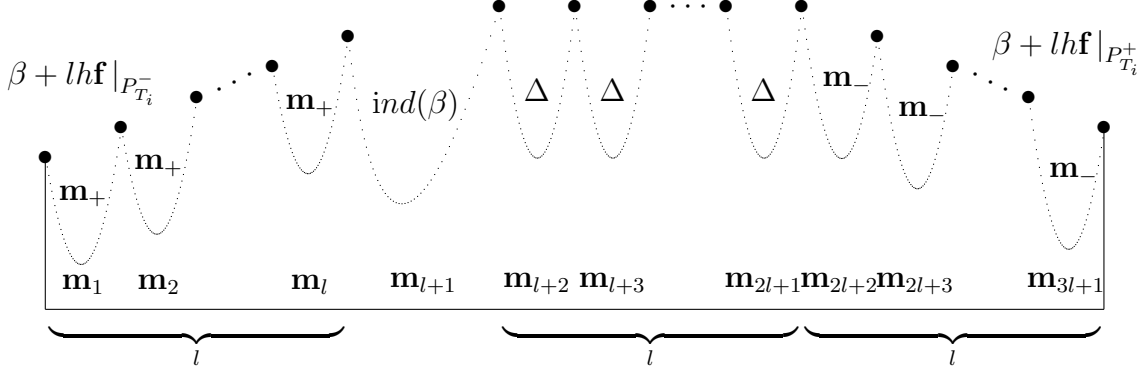


Figure 3: The concatenation of $3l+1$ paths in N_i .

since $\mathbf{f}(w_{\mathbf{c}_{t_j}}) > 0$. Now, according to Condition $\mathcal{T}.5$ and equation (1), we have

$$\begin{aligned}
\text{ind}(\mathbf{hf})|_{C(t_j)} &= \sum_{\mathbf{g} \in \{\mathbf{c}_{t_j}\} \cup \Gamma_{t_j}} \mathbf{hf}(w_{\mathbf{g}})\mathbf{g}|_{C(t_j)} \\
&= n_j k_j \left(\mathbf{f}(w_{\mathbf{c}_{t_j}})\mathbf{c}_{t_j} + \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{f}(w_{\mathbf{g}})\mathbf{g} \right) |_{C(t_j)} \\
&= n_j \left((k_j \mathbf{f}(w_{\mathbf{c}_{t_j}}) - 1)\mathbf{c}_{t_j} + (\mathbf{c}_{t_j} + k_j \mathbf{f}_j) \right) |_{C(t_j)} \in \mathbf{R}(t_j)^*.
\end{aligned} \tag{4}$$

The same holds (because of equation (3)) for

$$\begin{aligned}
\text{ind}(\mathbf{hf} + \mathbf{e})|_{C(t_j)} &= \sum_{\mathbf{g} \in \{\mathbf{c}_{t_j}\} \cup \Gamma_{t_j}} (\mathbf{hf} + \mathbf{e})(w_{\mathbf{g}})\mathbf{g}|_{C(t_j)} = \\
&\left(n_j ((k_j \mathbf{f}(w_{\mathbf{c}_{t_j}}) - 1)\mathbf{c}_{t_j} + (\mathbf{c}_{t_j} + k_j \mathbf{f}_j)) + \mathbf{e}(w_{\mathbf{c}_{t_j}})\mathbf{c}_{t_j} + \mathbf{e}_j \right) |_{C(t_j)} = \\
&(l_j \mathbf{c}_{t_j} + (n_j - 1)(\mathbf{c}_{t_j} + k_j \mathbf{f}_j) + (\mathbf{c}_{t_j} + k_j \mathbf{f}_j + \mathbf{e}_j)) |_{C(t_j)} \in \mathbf{R}(t_j)^*.
\end{aligned} \tag{5}$$

Analogously, we have $\text{ind}(\mathbf{hf} + \mathbf{c}_t)|_{C(t_j)} \in \mathbf{R}(t_j)^*$ and $\text{ind}(\mathbf{hf} + \mathbf{d})|_{C(t_j)} \in \mathbf{R}(t_j)^*$, and by combining all transitions in T_i (like those in equations (4) and (5)), we get

$$\text{ind}(2\mathbf{hf} + \mathbf{c}_t)|_{C(T_i)}, \text{ind}(2\mathbf{hf} + \mathbf{c}_t + \mathbf{e})|_{C(T_i)}, \Delta \in \mathbf{R}(N_i).$$

for $\Delta := \text{ind}(\mathbf{hf})|_{C(T_i)} + \mathbf{d}$. (It holds $\forall p \in P_{T_i} \Delta_+(p) - \Delta(p^-) = \Delta_-(p) - \Delta(p^+)$.) Now, we will show that for every $\beta \in \mathbf{c}_t + \Gamma_t^*$ with $\text{ind}(\beta)|_{C(T_i)} \in \mathbf{R}(N_i)$, there exists a sufficiently large $l \geq 0$ such that there are $\mathbf{m}_\mu \in \mathbf{R}(N_i)$ for all $1 \leq \mu \leq 3l + 1$ fulfilling the following conditions:

It holds for all $1 \leq \mu \leq l$ and all $p \in P_{T_i}$

$$\begin{aligned}
\mathbf{m}_\mu(p^-) &= \beta(p^-) + (l - \mu + 1)h\mathbf{f}(p^-) + (\mu - 1)\Delta_+(p) \\
\mathbf{m}_\mu(p^+) &= \beta(p^-) + (l - \mu)h\mathbf{f}(p^-) + \mu\Delta_+(p) \\
\mathbf{m}_{l+1}(p^-) &= \beta(p^-) + l\Delta_+(p) \\
\mathbf{m}_{l+1}(p^+) &= \beta(p^+) + l\Delta_+(p) \\
\mathbf{m}_{l+1+\mu}(p^-) &= \beta(p^+) + (l - \mu + 1)\Delta_+(p) + (\mu - 1)\Delta^-(p) \\
\mathbf{m}_{l+1+\mu}(p^+) &= \beta(p^+) + (l - \mu)\Delta_+(p) + \mu\Delta^-(p) \\
\mathbf{m}_{2l+1+\mu}(p^-) &= \beta(p^+) + (l - \mu + 1)\Delta_-(p) + (\mu - 1)h\mathbf{f}(p^+) \\
\mathbf{m}_{2l+1+\mu}(p^+) &= \beta(p^+) + (l - \mu)\Delta_-(p) + \mu h\mathbf{f}(p^+).
\end{aligned}$$

Since, according to equation (2), $\Delta_+(p) > 0$ and $h\mathbf{f}(p^-) \geq 0$, it holds $\mathbf{m}_\mu - \mathbf{m}_+ \in \mathbf{Id}_{P_{T_i}}$ for a sufficiently large l . Together with $\mathbf{m}_+ \in \mathbf{R}(N_i)$, according to Condition $\mathcal{T}.4$, it follows that $\mathbf{m}_\mu \in \mathbf{m}_+ + \mathbf{Id}_{P_{T_i}} \subseteq \mathbf{R}(N_i)$ for all $1 \leq \mu \leq l$ and, analogously, $\mathbf{m}_{2l+1+\mu} \in \mathbf{m}_- + \mathbf{Id}_{P_{T_i}} \subseteq \mathbf{R}(N_i)$. According to Condition $\mathcal{T}.2$, it holds $\mathbf{m}_{l+1} \in \text{ind}(\beta)|_{C(T_i)} + \mathbf{Id}_{P_{T_i}} \subseteq \mathbf{R}(N_i)$ and, analogously, $\mathbf{m}_{l+1+\mu} \in \Delta + \mathbf{Id}_{P_{T_i}} \subseteq \mathbf{R}(N_i)$.

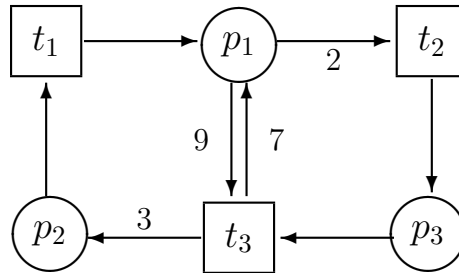
Since $\mathbf{m}_\mu(p^+) = \mathbf{m}_{\mu+1}(p^-)$ for all $1 \leq \mu \leq 3l$ and all $p \in P_{T_i}$, we can concatenate all the \mathbf{m}_μ 's to one $\mathbf{m} \in \mathbf{R}(N_i)$ with $\text{ind}(\beta + lh\mathbf{f})|_{C(T_i)} \in \mathbf{m} + \mathbf{Id}_{P_{T_i}} \subseteq \mathbf{R}(N_i)$ and $\mathbf{m}|_{P_{T_i}^- \cup P_{T_i}^+} = (\beta + lh\mathbf{f})|_{P_{T_i}^- \cup P_{T_i}^+}$; thus,

$$(\beta + lh\mathbf{f})|_{C(t)} \in (\mathbf{c}_t + \Gamma_t^*) \hat{\circ}_{A_t} \sum_{N_i \in K_t} \mathbf{R}(N_i) = \mathbf{R}(t)$$

for $A_t = \bigcup_{N_i \in K_t} C(N_i)$. For $k := 2h + lh$ and $\beta = 2h\mathbf{f} + \mathbf{c}_t$ or $\beta = 2h\mathbf{f} + \mathbf{c}_t + \mathbf{e}$ we get $(\mathbf{c}_t + k\mathbf{f})|_{C(t)}, (\mathbf{c}_t + k\mathbf{f} + \mathbf{e})|_{C(t)} \in \mathbf{R}(t)$. \blacksquare

Example:

Consider a transition t , where K_t contains only the following sub-net:



This means $\mathbf{c}_{t_1} = \begin{bmatrix} p_2^- & w_1 & p_1^+ \\ 1 & 1 & 1 \end{bmatrix}$, $\mathbf{c}_{t_2} = \begin{bmatrix} p_1^- & w_2 & p_3^+ \\ 1 & 1 & 1 \end{bmatrix}$, $\mathbf{c}_{t_3} = \begin{bmatrix} p_1^- & p_3^- & w_3 & p_1^+ & p_2^+ \\ 9 & 1 & 1 & 7 & 3 \end{bmatrix}$ and $K_{t_1} = K_{t_2} = K_{t_3} = \emptyset$. Let, furthermore, $\Gamma_t = \{\mathbf{g}_1, \mathbf{g}_2\}$ with

$$\mathbf{g}_1 = \begin{bmatrix} p_2^- & w_1 & w_2 & p_3^+ \\ 2 & 2 & 1 & 1 \end{bmatrix}, \quad \mathbf{g}_2 = \begin{bmatrix} p_2^- & w_1 & w_2 & w_3 \\ 1 & 4 & 1 & 1 \end{bmatrix}, \quad \mathbf{c}_t = \begin{bmatrix} p_3^- & w_1 & w_3 & p_2^+ \\ 1 & 2 & 1 & 1 \end{bmatrix}.$$

Comparing $\text{ind}(\mathbf{c}_t) = \begin{bmatrix} p_1^- & p_2^- & p_3^- & w_1 & w_3 & p_1^+ & p_2^+ \\ 9 & 2 & 1 & 2 & 1 & 9 & 3 \end{bmatrix}$ with \mathbf{c}_t , we can see that Condition $\mathcal{T}.2$ is fulfilled, but \mathbf{c}_t does not provide enough tokens on p_1 to allow the

firing of t_3 . We choose

$$\begin{aligned}\mathbf{f} &= \mathbf{g}_1 + \mathbf{g}_2 = \left[\begin{array}{cccccc} p_2^- & w_1 & w_2 & w_3 & p_3^+ \\ 3 & 6 & 2 & 1 & 2 \end{array} \right], \\ \mathbf{m}_+ &= \left[\begin{array}{cccccc} p_2^- & w_1 & w_2 & p_1^+ & p_3^+ \\ 3 & 3 & 1 & 1 & 1 \end{array} \right], \\ \mathbf{m}_- &= \left[\begin{array}{cccccc} p_1^- & p_2^- & w_1 & w_2 & p_3^+ \\ 1 & 1 & 1 & 1 & 2 \end{array} \right]\end{aligned}$$

and $h = 2$ is large enough for $\Delta_+ = \left[\begin{array}{ccc} p_1 & p_2 & p_3 \\ 1 & 3 & 1 \end{array} \right]$ and $\Delta_- = \left[\begin{array}{ccc} p_1 & p_2 & p_3 \\ 1 & 1 & 1 \end{array} \right]$. Looking at

$$\begin{aligned}\beta &= \mathbf{c}_t + 2h\mathbf{f} = \left[\begin{array}{cccccc} p_2^- & p_3^- & w_1 & w_2 & w_3 & p_2^+ & p_3^+ \\ 6 & 1 & 14 & 4 & 3 & 1 & 2 \end{array} \right] \text{ and} \\ \text{ind}(\beta) &= \left[\begin{array}{cccccc} p_1^- & p_2^- & p_3^- & w_1 & w_2 & w_3 & p_1^+ & p_2^+ & p_3^+ \\ 35 & 14 & 3 & 14 & 4 & 3 & 35 & 9 & 4 \end{array} \right]\end{aligned}$$

we can see that $l = 35$ is sufficient. This also suffices for $\Delta = \text{ind}(h\mathbf{f}) - \mathbf{m}_+ - \mathbf{m}_- =$

$$\left[\begin{array}{cccccc} p_1^- & p_2^- & p_3^- & w_1 & w_2 & w_3 & p_1^+ & p_2^+ & p_3^+ \\ 25 & 8 & 2 & 8 & 2 & 2 & 25 & 6 & 1 \end{array} \right].$$

This means for $k = 2h + lh = 74$ we get $(c_t + 74(g_1 + g_2))|_{C(t)} \in \mathbf{R}(t)$.

6 The reachability relation for Petri nets with inhibitor arcs

Now, we generalize Lemma 2.3 by using the operators \cup , \circ_Q and \star_Q over finite sets of multisets in a nested way. This allows us to express the reachability problem in a Petri net for which there exists an ordering of the places such that a place has an inhibitor arc to all those transitions which have an inhibitor arc from a preceding place:

Theorem 6.1 *In a Petri-net $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ with*

$$\exists \mathbf{g} \in \mathbb{N}_+^P \forall p, p' \in P \mathbf{g}(p) \leq \mathbf{g}(p') \rightarrow (\forall t \in T (p', t) \in I \rightarrow (p, t) \in I),$$

we can construct an expression T_g such that there is a firing sequence $w \in T^$ with $\mathbf{m}_0[w]\mathbf{m}_e$ if and only if $\mathbf{R}(T_g)$ is $(= \{\emptyset\})$ and) not empty.*

With Theorem 3.1 we derive the following:

Corollary 6.1 *The reachability problem for a Petri net $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ with*

$$\exists \mathbf{g} \in \mathbb{N}_+^P \forall p, p' \in P \mathbf{g}(p) \leq \mathbf{g}(p') \rightarrow (\forall t \in T (p', t) \in I \rightarrow (p, t) \in I),$$

is decidable.

Proof: (of Theorem 6.1) Let the Petri-net again have the properties of lemmata 2.1 and 2.2. Let $P_{T_h} = \{p \mid \mathbf{g}(p) \geq h\}$ be the places accessible on level h ; this level can only represent markings having no token on a place p with $\mathbf{g}(p) < h$. The innermost expression T_1 is given by

$$T_1 := \{t \mid t \in T, \forall p \in P(p, t) \notin I\}$$

describing transitions having no inhibitor arc. In general, the expression T_h on level $h > 1$ is given by

$$T_h := \{t_h\} \cup \{t \mid t \in T, \forall p \in P \mathbf{g}(p) \geq h \rightarrow (p, t) \notin I \wedge \forall p \in P \mathbf{g}(p) < h \rightarrow W(p, t) = W(t, p) = 0\}$$

with $t_h = t_{P \setminus P_{T_h}}(\{\star_{P_{T_{h-1}}}(T_{h-1})\})$ in accordance with Lemma 3.1.

On the top level $g = \max\{\mathbf{g}(p) \mid p \in P\} + 1$, we have

$$T_g := \{t_g\} := \{(\mathbf{m}_0^- + \mathbf{m}_e^+) \hat{\circ}_{P_{T_{g-1}}^+ \cup P_{T_{g-1}}^-} \star_{P_{T_{g-1}}} (T_{g-1})\}.$$

Now, we have to show that $\exists w \in T^* m_0[w]m_e$ if and only if $\mathbf{R}(T_g) \neq \emptyset$:

The firing sequence w can be decomposed in minimal firing sequences $w_1 \dots w_l$ having the property $\mathbf{m}_0[w_1]\mathbf{m}_1[w_2] \dots [w_l]\mathbf{m}_k = \mathbf{m}_e$ such that $\mathbf{m}_i(p) = 0$ for all $i \leq l$ and p with $\mathbf{g}(p) < g - 1$.

In a general induction step for $h < g - 1$, we are given a firing sequence $\mathbf{m}'_0[w']\mathbf{m}'_e$. It starts and ends with a marking without a token on a place p with $\mathbf{g}(p) \leq h$. However, intermediately there is always a token on a place p with $\mathbf{g}(p) \leq h$ in the markings. This sequence w' can be decomposed into minimal firing sequences $w_1 \dots w_k$ having the property $\mathbf{m}'_0[w_1]\mathbf{m}'_1[w_2] \dots [w_k]\mathbf{m}'_k = \mathbf{m}'_e$ such that $\mathbf{m}'_i(p) = 0$ for all $i \leq k$ and p with $\mathbf{g}(p) < h$. Thus, for all $1 < i < k$, there is a p with $\mathbf{g}(p) = h$ and $\mathbf{m}'_i(p) > 0$.

If $w_i = t_i \in T$ then $W(p, t_i) = W(t_i, p) = 0$ for all p with $\mathbf{g}(p) < h$ and $(p, t_i) \notin I$ for all p with $\mathbf{g}(p) \geq h$. Thus, $t_i \in T_h$ with $K_{t_i} = \emptyset$; therefore,

$$\mathbf{m}'_{i-1}^- + \mathbf{m}'_i^+ = \{p^- \mapsto \mathbf{m}'_{i-1}(p), p^+ \mapsto \mathbf{m}'_i(p) \mid p \in P\} \in \mathbf{R}(T_h).$$

(For $h = 1$, this is the only case, and this starts the induction.) Otherwise, by minimality of w_i , there is always a token on a place p with $\mathbf{g}(p) < h$ in the intermediate markings. Thus, by induction over h , it holds

$$\mathbf{m}'_{i-1}^- + \mathbf{m}'_i^+ \in \mathbf{R}(\star_{P_{T_{h-1}}} (T_{h-1})) = \mathbf{R}(t_h) \subseteq \mathbf{R}(T_h)$$

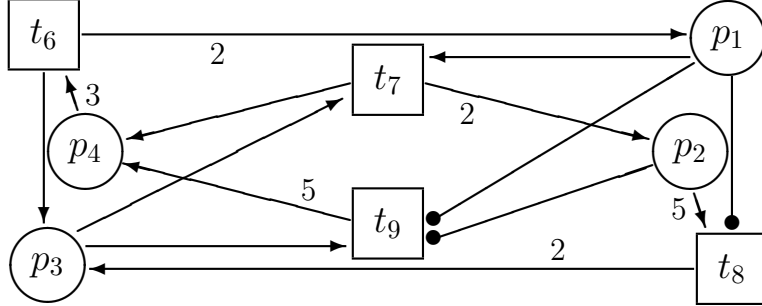
as well. This means $\mathbf{m}'_0^- + \mathbf{m}'_e^+ \in \mathbf{R}(\star_{P_{T_h}} (T_h))$, which completes the induction.

On the top level, by concatenation of all $\mathbf{m}'_{i-1}^- + \mathbf{m}'_i^+ \in \mathbf{R}(\star_{P_{T_{g-1}}} (T_{g-1}))$ for all $1 \leq i \leq g$, we analogously, get

$$(\mathbf{m}_0^- + \mathbf{m}_e^+) \in \mathbf{R}(\star_{P_{T_{g-1}}} (T_{g-1})); \text{ thus, } \emptyset \in \mathbf{R}(T_g).$$

The other direction again follows simply by composing firing sequences. ■

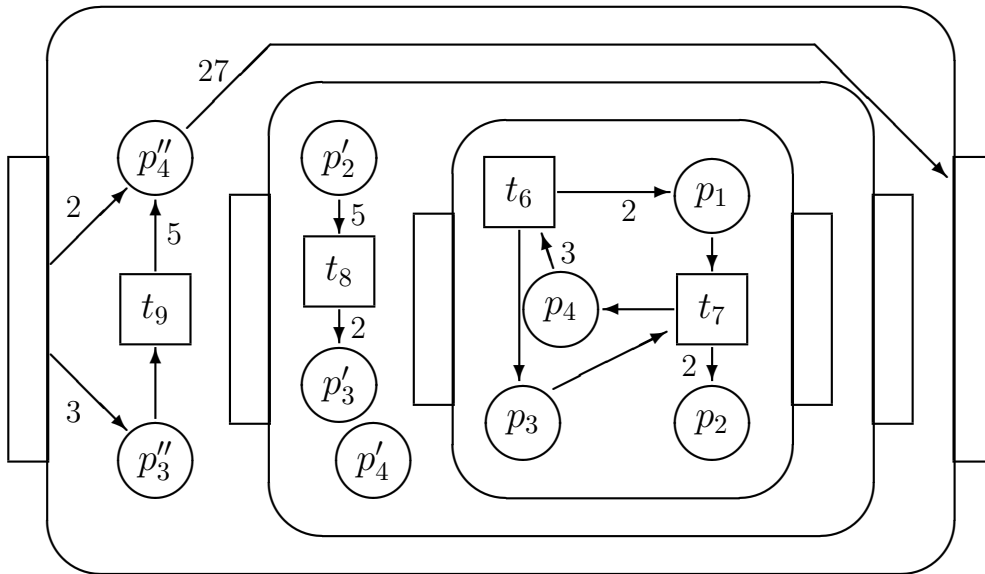
Example: The start marking $\{p_3 \mapsto 3, p_4 \mapsto 2\}$ and the end marking $\{p_4 \mapsto 27\}$ of the Petri net



with the function \mathbf{g} with $\mathbf{g}(p_1) = 1$, $\mathbf{g}(p_2) = 2$ and $\mathbf{g}(p_3) = \mathbf{g}(p_4) = 3$ leads to

$$T_1 = \left\{ \begin{bmatrix} p_4^- & p_1^+ & p_3^+ \\ 3 & 2 & 1 \end{bmatrix}, \begin{bmatrix} p_1^- & p_3^- & p_2^+ & p_4^+ \\ 1 & 1 & 2 & 1 \end{bmatrix} \right\} + \mathbf{Id}_P.$$

This enables the firing sequence $w = t_6 t_7 t_7$ from $\begin{bmatrix} p_3 \\ 1 \end{bmatrix}, \begin{bmatrix} p_4 \\ 3 \end{bmatrix}$ to $\begin{bmatrix} p_2 \\ 4 \end{bmatrix}, \begin{bmatrix} p_4 \\ 2 \end{bmatrix}$ on the innermost level as $\begin{bmatrix} p_3^- & p_4^- & p_2^+ & p_4^+ \\ 1 & 3 & 4 & 2 \end{bmatrix} \in \mathbf{R}(\star_{P_{T_1}}(T_1)) = \mathbf{R}(t_2) \subseteq \mathbf{R}(T_2)$. Together with $\begin{bmatrix} p_2^- & p_3^+ \\ 5 & 2 \end{bmatrix} \in \mathbf{R}(T_2)$ for t_8 , we get the firing sequence $w' = (w)(w)t_8(w)t_8(w)t_8(w)t_8$ from $\begin{bmatrix} p_3 \\ 2 \end{bmatrix}, \begin{bmatrix} p_4 \\ 7 \end{bmatrix}$ to $\begin{bmatrix} p_3 \\ 5 \end{bmatrix}, \begin{bmatrix} p_4 \\ 2 \end{bmatrix}$ on the next level as $\begin{bmatrix} p_3^- & p_4^- & p_3^+ & p_4^+ \\ 2 & 7 & 5 & 2 \end{bmatrix} \in \mathbf{R}(\star_{P_{T_2}}(T_2)) = \mathbf{R}(t_3) \subseteq \mathbf{R}(T_3)$. Together with $\begin{bmatrix} p_3^- & p_4^+ \\ 1 & 5 \end{bmatrix} \in \mathbf{R}(T_3)$ for t_9 , this enables the firing sequence $w'' = t_9(w')t_9^5$ from $\begin{bmatrix} p_3 \\ 3 \end{bmatrix}, \begin{bmatrix} p_4 \\ 2 \end{bmatrix}$ to $\begin{bmatrix} p_4 \\ 27 \end{bmatrix}$ on the following level as $\begin{bmatrix} p_3^- & p_4^- & p_4^+ \\ 3 & 2 & 27 \end{bmatrix} \in \mathbf{R}(\star_{P_{T_3}}(T_3)) = \mathbf{R}(t_4) = \mathbf{R}(T_4)$.



7 Priority-Multicounter-Automata

We define a priority-multicounter-automaton by a restrictive zero-test according to an order of the counters in the following way: the first counter can be tested for zero at any time; the second counter can only be tested for zero simultaneously with the first counter; any further counter can only be tested for zero simultaneously with all preceding counters. Formally, this reads as follows:

A *priority-multicounter-automaton* is a one-way automaton described by the 6-tuple

$$A = (k, Z, \Sigma, \delta, z_0, E)$$

with the set of states Z , the input alphabet Σ , the transition relation

$$\delta \subseteq (Z \times (\Sigma \cup \{\lambda\}) \times \{0 \dots k\}) \times (Z \times \{-1, 0, 1\}^k),$$

initial state z_0 , the accepting states $E \subseteq Z$, the set of configurations $C_A = Z \times \Sigma^* \times \mathbb{N}^k$, the initial configuration $\sigma_A(x) = \langle z_0, x, \underbrace{0, \dots, 0}_k \rangle$ and configuration

transition relation

$$\langle z, ax, n_1, \dots, n_k \rangle \vdash_A \langle z', x, n_1 + i_1, \dots, n_k + i_k \rangle$$

if and only if $z, z' \in Z, a \in \Sigma \cup \{\lambda\}, \langle (z, a, j), (z', i_1, \dots, i_k) \rangle \in \delta, \forall i \leq j, n_i = 0$.

The language recognized by an priority-multicounter-automaton A is $L(A) = \{w \mid \exists z_e \in E \exists n_1, \dots, n_k \in \mathbb{N} \langle z_0, w, 0, \dots, 0 \rangle \vdash_A^* \langle z_e, \lambda, n_1, \dots, n_k \rangle\}$. A priority-multicounter-automaton can be changed in such a way that it has only one accepting state z_e and that all counters are empty while accepting. Thus, $L(A) = \{w \mid \langle z_0, w, 0, \dots, 0 \rangle \vdash_A^* \langle z_e, \lambda, 0, \dots, 0 \rangle\}$.

Using Theorem 6.1, we show that the emptiness problem of the accepted language is decidable for priority-multicounter-automata. The same holds for the halting problem by constructing an automaton which contains its input in the states.

Theorem 7.1 *The emptiness problem for priority-multicounter-automata is decidable.*

Proof: Given A we construct a Petri net (P, T, W, I, m_0, m_1) with the places

$P := \{1 \dots k\} \cup Z$, the transitions $T = \delta$, the weights W with

$W(z, ((z', a, j), (z'', V))) := 1$ if $z = z'$ else $:= 0$;

$W(((z', a, j), (z'', V)), z) := 1$ if $z = z''$ else $:= 0$;

$W(i, ((z', a, j), (z'', V))) := 1$ if $V(i) = -1$ else $:= 0$; and

$W(((z', a, j), (z'', V)), i) := 1$ if $V(i) = 1$ else $:= 0$;

the inhibitor arcs $I := \{(i, ((z', a, j), (z'', V))) \mid i \leq j\}$, the start marking $\mathbf{m}_0 := \{z_0 \mapsto 1\}$, and the end marking $\mathbf{m}_1 := \{z_e \mapsto 1\}$ which is reachable from \mathbf{m}_0 if and only if $L(A) \neq \emptyset$. According to Corollary 3.1 with $\mathbf{g}(i) = i$ for $i \leq k$ and $\mathbf{g}(z) = k + 1$ for $z \in Z$, this is decidable. ■

Consequences of this result for the decidability of the synchronizability and maximality of semi-trace languages are formulated in [Rei94]. Further consequences are described in [FFOR05]: Every unary language is recursive if it is generated by a graph-controlled grammar with an arbitrary number of nonterminal symbols but only one of the nonterminal symbols being allowed to be used in the appearance checking mode. Theorem 7.1 implies the optimality the results in [FFOR05] proving the computational completeness of graph-controlled grammars, programmed grammars, and matrix grammars with a certain nonterminal complexity.

The classes k -PMC of languages accepted by a priority-multicounter-automaton with $k > 0$ counters (and also their union) are incomparable to the class LIN of linear languages and it holds $(k-1)$ -PMC \subsetneq k -PMC. This is because

$$\{a^{n_1}ba^{n_2}\dots ba^{n_{k+1}}\$a^{n_{k+1}}b\dots a^{n_2}ba^{n_1} \mid \forall i \leq k+1 \ n_i \in \mathbb{N}\} \notin k\text{-PMC}.$$

This can be shown by constructing T fulfilling property \mathcal{T} and, then, by using Lemma 4.1 to find two different words in the language where the automaton has the same configuration reading $\$$. With the same argument, this also holds for the classes k -BLIND and k -PBLIND in [Gre78]. Furthermore, $\{(a^n b)^m \mid n, m \in \mathbb{N}\}$ cannot be accepted by a priority-multicounter-automaton (Theorem 3.2 in [Rei94]).

8 Restricted Priority- Multipushdown- Automata

We define a priority-multipushdown-automaton by a different treatment of one of the two pushdown symbols according to an order of the pushdown stores in the following way: let the pushdown alphabet be $\{0, 1\}$. A 0 can be pushed to and popped from every pushdown store independently, but a 1 can only be pushed to or popped from a pushdown store if all pushdown stores with a lower order are empty. Furthermore, the restriction requires that if a 1 is popped from a pushdown store, then a 1 cannot be pushed anymore to this store until it is empty.

Theorem 8.1 *The emptiness problem for restricted priority-multipushdown-automata is decidable.*

This generalizes the result in [JKLP90] that $\underline{LIN}\%D_1^*$ (the class of languages generated by linear grammar and deletion of semi Dyck words) is recursive. We conjecture that decidability still holds in the unrestricted case but, even in the special case of a pushdown automaton with additional weak counters (without zero-test), this is still an open problem.

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Appendix

A Alternative proof of Corollary 3.1

Alternatively, instead of using Lemmata 2.3, 2.1 and 2.2, we can construct T for a given net $N = (P, T, W, \{(p_1, \hat{t})\}, \mathbf{m}_0, \mathbf{m}_e)$ directly as follows: Let $N' := \mathbf{K}_P(T \setminus \{\hat{t}\})$ be the expression for firing sequences $w \in (T \setminus \{\hat{t}\})^*$ and \hat{N}' be the result of replacing all places like in the the proof of Lemma 3.1. No we construct $T_2 := (c_2 + \Gamma_2^*) \hat{\circ}_{A_t} \hat{N}'$ with

$$\begin{aligned} c_2 &:= c_{\hat{t}}^- + \{\hat{p}^+ \mapsto c_i(p^+) \mid p \in P\} \text{ and} \\ \Gamma_2 &:= \{\{\hat{p}^s \mapsto 1, p^s \mapsto 1\} \mid p \in P \setminus \{p_1\}, s \in \{+, -\}\} \end{aligned}$$

which expresses those firing sequences starting with \hat{t} and ending with markings without tokens on p_1 . and analogously $T_e :=$

$$(c_{\hat{t}}^- + \{\hat{p}^+ \mapsto c_i(p^+) \mid p \in P\} + \mathbf{m}_e^+) + \{\{\hat{p}^- \mapsto 1, p^- \mapsto 1\} \mid p \in P \setminus \{p_1\}\}^* \hat{\circ}_{A_t} \hat{N}'$$

which expresses those firing sequences starting with \hat{t} and ending with m_e . Using Lemma 3.4 and other reformulations, we can construct

$$T := \mathbf{m}_0^+ \circ_P N' \circ_{P \setminus \{p_1\}} \mathbf{K}_{P \setminus \{p_1\}}(T_2) \circ_{P \setminus \{p_1\}} T_e \cup \mathbf{m}_0^+ \circ_P N' \circ_P \mathbf{m}_e^-.$$

Then, we construct T' according to Theorem 3.1. According to Condition 5 of property \mathcal{T} , $\mathbf{R}(T) = \mathbf{R}(T')$ is empty if and only if $T' = \emptyset$.

B Alternative proof of Lemma 4.1

Definition An expression T has the property \mathcal{T}' if T has the property \mathcal{T} in which Condition $\mathcal{T}.4$ is replaced by the following Condition $\mathcal{T}.4'$: $\forall p \in (P_{T_i}^+ \cup P_{T_i}^-) \sum_{g \in \Gamma_t} \mathbf{g}(p) > 0$.

Remark: $\mathcal{T}.4'$ and $\mathcal{T}.3$ together mean $\forall a \in C(N_i) \sum_{g \in \Gamma_t} \mathbf{g}(a) > 0$.

Lemma B.1 *If the conditions $\mathcal{T}.1 - \mathcal{T}.4$ hold for t , then we can construct a t' with $\mathbf{R}(t') = \mathbf{R}(t)$ such that conditions $\mathcal{T}.1 - \mathcal{T}.4'$ hold for t' and the size only increases in the last component.*

Proof: It follows from Condition $\mathcal{T}.4$ that there exists a sufficiently large $\mathbf{h} \in \Gamma_t^*$ such that for all i

$$\begin{aligned} \forall p \in P_{T_i} \quad & \mathbf{d}(p^-) := \mathbf{h}(p^-) - \mathbf{m}_+(p^-) + \mathbf{m}_+(p^+) \geq 1 \wedge \\ & \mathbf{d}(p^+) := \mathbf{h}(p^+) + \mathbf{m}_-(p^-) - \mathbf{m}_-(p^+) \geq 1 \wedge \\ & (\mathbf{c}_t + \mathbf{h})(p^-) \geq \mathbf{m}_+(p^-) \wedge (\mathbf{c}_t + \mathbf{h})(p^+) \geq \mathbf{m}_-(p^+), \end{aligned}$$

and, additionally, according to Condition $\mathcal{T}.3$,

$$\forall w \in C(N_i) \setminus (P_{T_i}^+ \cup P_{T_i}^-) \quad \mathbf{d}(w) := \mathbf{h}(w) - \mathbf{m}_+(w) - \mathbf{m}_-(w) \geq 1.$$

$$\text{otherwise } \forall a \in C(t) \quad \mathbf{d}(a) := \mathbf{h}(a)$$

Now, we can define t' with $K_{t'} = K_t$, $c_{t'} = c_t$ and $\Gamma_{t'} = \Gamma_t \cup \{\mathbf{d}\}$. Conditions 1 and 3 remain unchanged. Condition 2 still holds because for all $p \in P_{T_i}$

$$\begin{aligned} \mathbf{d}(p^-) - \text{ind}(\mathbf{d})(p^-) &= \\ \mathbf{h}(p^-) - \mathbf{m}_+(p^-) + \mathbf{m}_+(p^+) - (\text{ind}(\mathbf{h})(p^-) - \text{ind}(\mathbf{m}_+)(p^-) - \text{ind}(\mathbf{m}_-)(p^-)) &= \\ \mathbf{h}(p^-) - \text{ind}(\mathbf{h})(p^-) + \text{ind}(\mathbf{m}_+)(p^+) + \text{ind}(\mathbf{m}_-)(p^-) &= \\ \mathbf{h}(p^+) - \text{ind}(\mathbf{h})(p^+) + \text{ind}(\mathbf{m}_+)(p^+) + \mathbf{m}_-(p^-) - \mathbf{m}_-(p^+) + \text{ind}(\mathbf{m}_-)(p^+) &= \\ \mathbf{d}(p^+) - \text{ind}(\mathbf{d})(p^+) &= \end{aligned}$$

because the equation in Condition 2 was already fulfilled by \mathbf{h} , \mathbf{m}_+ and \mathbf{m}_- . Condition 4' holds according to the definition of d , and it holds $\mathbf{R}(t) \subseteq \mathbf{R}(t')$ since $\mathbf{L}_t \subseteq \mathbf{L}_{t'}$. So what remains is to show that $\mathbf{R}(t') \subseteq \mathbf{R}(t)$:

Let $\mathbf{m}' = \mathbf{m} + l\mathbf{d} \in \mathbf{L}_{t'}$ with $\mathbf{m} \in \mathbf{L}_t$ and $\mathbf{m}'|_{C(t)} \in \mathbf{R}(t')$, then for every $T_i \in K_t$, there are $\mathbf{m}_\mu \in \mathbf{R}(N_i)$ for all $1 \leq \mu \leq 2l + 1$ such that for all $p \in P_{T_i}$

$$\begin{aligned} \mathbf{m}_\mu(p^-) &= \mathbf{m}(p^-) + (l - \mu + 1)\mathbf{h}(p^-) + (\mu - 1)\mathbf{d}(p^-) \\ \mathbf{m}_\mu(p^+) &= \mathbf{m}(p^+) + (l - \mu)\mathbf{h}(p^+) + \mu\mathbf{d}(p^+) \\ \mathbf{m}_{l+1}(p^-) &= \mathbf{m}'(p^-) = \mathbf{m}(p^-) + l\mathbf{d}(p^-) \\ \mathbf{m}_{l+1}(p^+) &= \mathbf{m}'(p^+) = \mathbf{m}(p^+) + l\mathbf{d}(p^+) \\ \mathbf{m}_{l+1+\mu}(p^-) &= \mathbf{m}(p^-) + (l - \mu + 1)\mathbf{h}(p^-) + (\mu - 1)\mathbf{d}(p^-) \\ \mathbf{m}_{l+1+\mu}(p^+) &= \mathbf{m}(p^+) + (l - \mu)\mathbf{h}(p^+) + \mu\mathbf{d}(p^+) \end{aligned}$$

with $\text{ind}(\mathbf{m}')|_{C(T_i)} \in \mathbf{m}_{l+1} + \mathbf{Id}_{P_{T_i}}$, $\mathbf{m}_\mu \in \mathbf{m}_+ + \mathbf{Id}_{P_{T_i}} \subseteq \mathbf{R}(N_i)$ for all $1 \leq \mu \leq l$, and, analogously, $\mathbf{m}_{l+1+\mu} \in \mathbf{m}_- + \mathbf{Id}_{P_{T_i}} \subseteq \mathbf{R}(N_i)$.

Since $\mathbf{m}_\mu(p^+) = \mathbf{m}_{\mu+1}(p^-)$ for all $1 \leq \mu \leq 2l$ and all $p \in P_{T_i}$, we can concatenate all the \mathbf{m}_μ 's to $\text{ind}(\mathbf{m} + l\mathbf{h})|_{C(T_i)} \in \mathbf{R}(N_i)$ and, therefore, obtain $\mathbf{m}'|_{C(t)} = \mathbf{m} + l\mathbf{h}|_{C(t)} \in \mathbf{R}(t)$. \blacksquare

From this construction, it also follows that, in the proof of Lemma 4.1, we can choose an l such that $l\mathbf{f} - \mathbf{h} \geq \mathbf{f}$. We can then proof the lemma with $\mathbf{f}' = l\mathbf{f} - \mathbf{h} + \mathbf{d}$, under the assumption that conditions $\mathcal{T}.1 - \mathcal{T}.4'$ hold, and obtain $k = k'l$ by $k'\mathbf{f}'|_{C(t)} = k'l\mathbf{f}|_{C(t)}$.

Proof:(of Lemma 4.1) Given $\mathbf{f} \in \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g} + \Gamma_t^*$ and $\mathbf{e} \in (\Gamma_t \cup -\Gamma_t)^*$ we have to find

a $k \geq 2$ such that $\{(\mathbf{c}_t + k\mathbf{f})|_{C(t)}, (\mathbf{c}_t + k\mathbf{f} + \mathbf{e})|_{C(t)}\} \subseteq \mathbf{R}(t)$.

For an elementary transition t with $K_t = \emptyset$, we have $\mathbf{R}(t) = \mathbf{c}_t + \Gamma_t^*$ and the statement is easily fulfilled by choosing a sufficiently large k compensating negative components in e .

Induction step: For every $N_i \in K_t$ and for every $t_j \in T_i$, let

$$\mathbf{f}_j := \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{f}(w_{\mathbf{g}})\mathbf{g}, \quad \mathbf{e}_j := \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{e}(w_{\mathbf{g}})\mathbf{g} \quad \text{and} \quad \mathbf{h}_j := \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{c}_t(w_{\mathbf{g}})\mathbf{g}.$$

Since $\mathbf{f}(w_{\mathbf{g}}) > 0$ for every $\mathbf{g} \in \Gamma_{t_j}$, according to Condition $\mathcal{T}.3$, we have $\mathbf{f}_j \in \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{g} + \Gamma_{t_j}^*$. This means \mathbf{f}_j fulfills the condition for \mathbf{f} one level deeper.

When we use Condition $\mathcal{T}.1$ and apply the lemma by induction for sub-transitions t_j of t two times for \mathbf{e} as \mathbf{e}_j or \mathbf{h}_j and for \mathbf{f} as \mathbf{f}_j , we conclude that there exist $k_j, k'_j \geq 2$ with

$$\begin{aligned} &(\mathbf{c}_{t_j} + k_j \mathbf{f}_j)|_{C(t_j)}, (\mathbf{c}_{t_j} + k_j \mathbf{f}_j + \mathbf{e}_j)|_{C(t_j)}, \\ &(\mathbf{c}_{t_j} + k'_j \mathbf{f}_j)|_{C(t_j)}, (\mathbf{c}_{t_j} + k'_j \mathbf{f}_j + \mathbf{h}_j)|_{C(t_j)} \in \mathbf{R}(t_j). \end{aligned} \quad (1)$$

There exists a sufficiently large $h \geq 1$ with $h\mathbf{f} + \mathbf{e} \in \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g} + \Gamma_t^*$ such that, for all i and j , we have $h = n_j k_j = n'_j k'_j$ for some n_j, n'_j ,

$$\begin{aligned} l_j := n_j(k_j \mathbf{f}(w_{\mathbf{c}_{t_j}}) - 1) + \mathbf{e}(w_{\mathbf{c}_{t_j}}) &> 0 \text{ and} \\ n'_j(k'_j \mathbf{f}(w_{\mathbf{c}_{t_j}}) - 1) + \mathbf{c}_t(w_{\mathbf{c}_{t_j}}) &> 0 \end{aligned} \quad (3)$$

since $\mathbf{f}(w_{\mathbf{c}_{t_j}}) > 0$. Now we have

$$\begin{aligned} \text{ind}(h\mathbf{f})|_{C(t_j)} &= \sum_{\mathbf{g} \in \{\mathbf{c}_{t_j}\} \cup \Gamma_{t_j}} h\mathbf{f}(w_{\mathbf{g}})\mathbf{g}|_{C(t_j)} \\ &= n_j k_j \left(\mathbf{f}(w_{\mathbf{c}_{t_j}})\mathbf{c}_{t_j} + \sum_{\mathbf{g} \in \Gamma_{t_j}} \mathbf{f}(w_{\mathbf{g}})\mathbf{g} \right)|_{C(t_j)} \\ &= n_j \left((k_j \mathbf{f}(w_{\mathbf{c}_{t_j}}) - 1)\mathbf{c}_{t_j} + (\mathbf{c}_{t_j} + k_j \mathbf{f}_j) \right)|_{C(t_j)} \in \mathbf{R}(t_j)^* \end{aligned} \quad (4)$$

according to Condition $\mathcal{T}.5$ and equation (1). The same holds (because of equation (3)) for

$$\begin{aligned} \text{ind}(h\mathbf{f} + \mathbf{e})|_{C(t_j)} &= \sum_{\mathbf{g} \in \{\mathbf{c}_{t_j}\} \cup \Gamma_{t_j}} (h\mathbf{f} + \mathbf{e})(w_{\mathbf{g}})\mathbf{g}|_{C(t_j)} = \\ \left(n_j((k_j \mathbf{f}(w_{\mathbf{c}_{t_j}}) - 1)\mathbf{c}_{t_j} + (\mathbf{c}_{t_j} + k_j \mathbf{f}_j)) + \mathbf{e}(w_{\mathbf{c}_{t_j}})\mathbf{c}_{t_j} + \mathbf{e}_j \right)|_{C(t_j)} &= \\ \left(l_j \mathbf{c}_{t_j} + (n_j - 1)(\mathbf{c}_{t_j} + k_j \mathbf{f}_j) + (\mathbf{c}_{t_j} + k_j \mathbf{f}_j + \mathbf{e}_j) \right)|_{C(t_j)} &\in \mathbf{R}(t_j)^*. \end{aligned} \quad (5)$$

Analogously, we have $\text{ind}(h\mathbf{f} + \mathbf{c}_t)|_{C(t_j)} \in \mathbf{R}(t_j)^*$ and, by combination of all transitions in T_i (like those in equations (4) and (5)), we get

$$\text{ind}(h\mathbf{f})|_{C(T_i)}, \text{ind}(2h\mathbf{f} + \mathbf{c}_t)|_{C(T_i)}, \text{ind}(2h\mathbf{f} + \mathbf{c}_t + \mathbf{e})|_{C(T_i)} \in \mathbf{R}(N_i).$$

Since for all $p \in (P_{T_i}^+ \cup P_{T_i}^-)$ $\mathbf{f}(p) > 0$, we can find a sufficiently large l such that, by concatenation of $\text{ind}(2h\mathbf{f} + \mathbf{c}_t)|_{C(T_i)}$ respectively $\text{ind}(2h\mathbf{f} + \mathbf{c}_t + \mathbf{e})|_{C(T_i)}$ with $l - 2$ times $\text{ind}(h\mathbf{f})|_{C(T_i)}$, we have $(lh\mathbf{f} + \mathbf{c}_t)|_{C(t)}$, $(lh\mathbf{f} + \mathbf{c}_t + \mathbf{e})|_{C(t)} \in \mathbf{R}(t)$. \blacksquare

C Folding pushdown-stores into a nested Petri net

Formally, a *restricted priority-multipushdown-automaton* is a one-way automaton described by the 6-tuple

$$A = (k, Z, \Sigma, \delta, z_0, E)$$

with the set of states $Z = Z' \times \{\uparrow, \downarrow\}^k$, the input alphabet Σ , the transition relation

$$\delta \subseteq (Z \times (\Sigma \cup \{\lambda\}) \times \{0 \dots k\} \times \{\lambda, 0, 1\}^k) \times (Z \times \{\lambda, 0, 1\}^k),$$

initial state z_0 , the accepting states $E \subseteq Z$, the set of configurations $C_A = Z \times \Sigma^* \times (\{0, 1\}^k)^k$, the initial configuration $\sigma_A(x) = \langle z_0, x, 0^k \rangle$ and configuration transition relation

$$\langle z, ax, \mathbf{g}_1 d_1, \dots, \mathbf{g}_k d_k \rangle \xrightarrow{A} \langle z', x, \mathbf{g}_1 i_1, \dots, \mathbf{g}_k i_k \rangle$$

if and only if $z, z' \in Z, a \in \Sigma \cup \{\lambda\}, \langle (z, a, j, d_1, \dots, d_k), (z', i_1, \dots, i_k) \rangle \in \delta$, $z = (z'', a_1, \dots, a_k)$, $z' = (z''', a'_1, \dots, a'_k)$, $a_j = \uparrow \vee a'_j = \downarrow$, and

$$\begin{aligned} \forall i < j \quad \mathbf{g}_i &= \lambda \wedge \\ \forall i > j \quad d_i &\neq 1 \wedge i_i \neq 1 \wedge a_i = a'_i \wedge \\ \forall i \leq k \quad (a_i = \downarrow \vee d_i &\neq 1) \wedge (a'_i = \uparrow \vee i_i \neq 1). \end{aligned}$$

Furthermore, the condition $d_j \neq 0 \wedge i_j \neq 0$ can be established by creating an intermediate state and a smaller j in the second transition.

Proof:(of Theorem 8.1) Given A , which has without loss of generality, only one accepting configuration with all push-down stores empty and $\Sigma = \emptyset$, we add $|Z'|$ push-down stores playing the role of the states (Only one of them has a zero and the others are empty.). This allows us to set $Z = \{\uparrow, \downarrow\}^k$. Here, the end state becomes the last push-down store and the start state the second last; thus, without loss of generality the last 3 push-down stores never contain a 1. Then, we construct a nested Petri net on $2k - 3$ levels as follows:

Let $P_h := \{p_i \mid h < i \leq k\}$ and $P'_h := \{p_i, p'_i \mid h < i \leq k\}$. The innermost expression T_0 is

$$T_0 := \{t \mid K_t = \Gamma_t = \emptyset \wedge \exists \langle (z, \lambda, 0, d_1, \dots, d_k), (z, i_1, \dots, i_k) \rangle \in \delta \wedge \forall i \leq k \quad (\mathbf{c}_t(p_i^-) = 1 \leftrightarrow d_i = 0) \wedge (\mathbf{c}_t(p_i^+) = 1 \leftrightarrow i_i = 0) \}$$

which corresponds to pushing and popping only zeros. The net $N_0 = \star_{P_0}(T_0)$ is used twice in

$$t_1 = t(N_0) \circ_{\{(p_0^+, p_0^-), (p_0^-, p_0^+)\}} (t(N_0) \circ_Q \{ \{p_i^- \mapsto 1, p_i^{+'} \mapsto 1\} \{p_j^+ \mapsto 1, p_j^{-'} \mapsto 1\} \mid 0 < i \leq k \geq j \}^*)$$

with $Q := \{(p^-, p^-), (p^+, p^+) \mid p \in P_0\}$. This corresponds to a sequence pushing zeros on the first push-down store and a later sequence (on P'_0) popping the same number of zeros from the first push-down store. In general, net $N_{2h} = \star_{P_h}(T_{2h})$ is used twice in $t_{2h+1} =$

$$t(N_{2h}) \circ_{\{(p_h^+, p_h^-), (p_h^-, p_h^+)\}} (t(N_{2h}) \circ_Q \{\{p_i^- \mapsto 1, p_i^{+'} \mapsto 1\} \{p_i^+ \mapsto 1, p_i^{-'} \mapsto 1\} \mid h < i \leq k, h \leq j \leq k\}^*)$$

with $Q := \{(p^-, p^-), (p^+, p^+) \mid p \in P_h\}$. This corresponds to a sequence pushing zeros on the $h + 1$ -st push-down store and a later sequence (on P'_h) popping the same number of zeros from the first push-down store. This is used in $T_{2h-1} := \{t_{2h-1}\} \cup$

$$\{ t \mid K_t = \Gamma_t = \emptyset \wedge \\ \exists \langle (z, \lambda, h, d_1, \dots, d_k), (z, i_1, \dots, i_k) \rangle \in \delta \wedge \\ \exists \langle (z', \lambda, h, d'_1, \dots, d'_k), (z', i'_1, \dots, i'_k) \rangle \in \delta \wedge \\ i_h = d'_h = 1 \wedge i'_h = d_h = \lambda \wedge \\ \forall h < i \leq k (\mathbf{c}_t(p_i^-) = 1 \leftrightarrow d_i = 0) \wedge (\mathbf{c}_t(p_i^+) = 1 \leftrightarrow i_i = 0) \wedge \\ \forall h < i \leq k (\mathbf{c}_t(p_i^{+'}) = 1 \leftrightarrow d_i = 0) \wedge (\mathbf{c}_t(p_i^{-'}) = 1 \leftrightarrow i_i = 0) \},$$

which corresponds to pushing (respectively later simulated on P'_h popping) a one on the h -th push-down store.

Sequences in the net $N_{2h-1} = \star_{P'_{h-1}}(T_{2h-1})$ correspond to "folding" a pushing and a popping sequence together where the sequence on P' has reverse order. It appears in

$$t_{2h} = t(N_{2h-1}) \circ_Q \{\{p_i^+ \mapsto 1, p_i^{+'} \mapsto 1\} \{p_i^{-'} \mapsto 1, p_h^+ \mapsto 1\} \mid h < i \leq k\}^*)$$

with $Q := \{(p^+, p^+), (p^{+'}, p^{+'}), (p^{-'}, p^{-'}) \mid p \in P_h\}$. This matching of p_i^+ and $p_i^{+'}$ corresponds to the moment where the h -th push-down store switches from pushing to popping.

This is used in $T_{2h} := \{t_{2h}\} \cup$

$$\{ t \mid K_t = \Gamma_t = \emptyset \wedge \\ \exists \langle (z, \lambda, h+1, d_1, \dots, d_k), (z', i_1, \dots, i_k) \rangle \in \delta \wedge \\ z = (a_1, \dots, a_{h-1}, \downarrow, a_{h+1}, \dots, a_k), z' = (a'_1, \dots, a'_{h-1}, \uparrow, a_{h+1}, \dots, a_k), \wedge \\ \forall h < i \leq k (\mathbf{c}_t(p_i^-) = 1 \leftrightarrow d_i = 0) \wedge (\mathbf{c}_t(p_i^+) = 1 \leftrightarrow i_i = 0) \wedge \\ \forall h < i \leq k (\mathbf{c}_t(p_i^{+'}) = 1 \leftrightarrow d_i = 0) \wedge (\mathbf{c}_t(p_i^{-'}) = 1 \leftrightarrow i_i = 0) \}$$

which allows concatenating with the 0-test of the h -th push-down store in the net $N_{2h} = \star_{P_h}(T_{2h})$. On the top level $2k - 4$, we have $T_{2k-4} := \{t_{2k-4}\}$ with

$$t_{2k-4} = \{p_{k-1}^- \mapsto 1, p_k^+ \mapsto 1\} \circ_Q N_{2k-5}$$

with $Q := \{(p^-, p^-), (p^+, p^+) \mid p \in P'_h\}$.

Now, if we have a sequence $w \in \delta^*$ of transitions of A leading from the start configuration to an end configuration, then 1's are only pushed or popped from the first $k - 3$ push-down stores. We have to show that $\{p_{k-1}^- \mapsto 1, p_k^+ \mapsto 1\} \in \mathbf{R}(N_{2k-5})$ in order to obtain $\mathbf{R}(T_{2k-4}) = \{\emptyset\}$.

By induction over h , we consider $w \in \{\langle (z, \lambda, j, d_1, \dots, d_k), (z', i_1, \dots, i_k) \rangle \in \delta \mid j \leq h\}^*$ to be a sequence of transitions of A such that,

- in the corresponding sequence of configurations 1's are only pushed or popped from the first h push-down stores and
- they are empty in the first and the last configuration.

In this case, according to the definition, the direction of a_h cannot be changed from \downarrow to \uparrow . Thus, w can be decomposed into $v_1 t_1 v_2 t_2 \dots v_n w_n \dots s_2 w_2 s_1 w_1 = w$ such that t_i (s_i , respectively) with $i < n$ is a transition in δ with $j = h$ and $i_j = 1$ ($d_j = 1$, respectively); and the v_i and w_i are sequences of transitions in δ^* where no 1 is pushed or popped to the h 'th push-down store.

Each of the v_i or w_i can be decomposed into minimal sequences $w'_1 t'_1 w'_2 t'_2 \dots w'_m$. Here, each is starting and ending with the first $h - 1$ push-down stores empty and the t'_i are those transitions where the push-down store number $h - 1$ is switched from popping to pushing; that means $a_j = \uparrow \wedge a'_j = \downarrow$. The w'_i now have the same property as w with $h := h - 1$.

For $h = 0$ a sequence $w \in \{\langle (z, \lambda, 0, d_1, \dots, d_k), (z', i_1, \dots, i_k) \rangle \in \delta\}^*$ corresponds to an element in $\mathbf{R}(N_0)$.

By induction, we assume that, for every w'_i , we have a corresponding element in $\mathbf{R}(N_{2h-1})$ and, thus, in $\mathbf{R}(t_{2h})$. Furthermore, for every t'_i , we have a corresponding element in $\mathbf{R}(T_{2h})$. Thus, for every v_i and w_i , we have corresponding elements in $\mathbf{R}(N_{2h})$ which, together, yield a corresponding element in $\mathbf{R}(t_{2h+1})$. Furthermore, for every pair t_i, s_i , we have a corresponding element in $\mathbf{R}(T_{2h+1})$. Thus, for w , we have a corresponding path in $\mathbf{R}(N_{2h+1})$. This completes the induction.

In the other direction, if $\mathbf{R}(T_{2k-4}) \neq \emptyset$, composing the corresponding transitions in the appropriate way leads from the start to the end configuration. ■

D Definitions pure

Multisets: $\{b \mapsto f(b) \mid b \in B\} = \left[\begin{array}{c} b_1 \\ f(b_1), f(b_2), \dots, f(b_n) \end{array} \right] = \begin{pmatrix} \mathbf{f}(b_1) \\ \mathbf{f}(b_2) \\ \vdots \\ \mathbf{f}(b_n) \end{pmatrix} \in \mathbb{N}^B.$

Restriction: $\mathbf{f}|_A := \{b \mapsto \mathbf{f}(b) \mid b \in A\}$ $\mathbf{f}|_{\bar{A}} := \{b \mapsto \mathbf{f}(b) \mid b \notin A\}$ see page 4.

Star for a set \mathbf{M} of multisets: $\mathbf{M}^0 := \{\emptyset\}$, $\mathbf{M}^{i+1} := \mathbf{M}^i + \mathbf{M}$ and $\mathbf{M}^* := \bigcup_i \mathbf{M}^i.$

$$\mathbf{N} \circ_Q \mathbf{M} := \left\{ \mathbf{n} \Big|_{\pi_1(Q)} + \mathbf{m} \Big|_{\pi_2(Q)} \mid \mathbf{n} \in \mathbf{N}, \mathbf{m} \in \mathbf{M}, \forall (a, b) \in Q \mathbf{n}(a) = \mathbf{m}(b) \right\}.$$

$\hat{\circ}_A := \circ_{\{(a,a) \mid a \in A\}}$

Identity: $\mathbf{Id}_Q := \{\{a \mapsto 1, b \mapsto 1\} \mid (a, b) \in Q\}^*.$

For Q with $\pi_1(Q)$ and $\pi_2(Q)$ disjoint, we define $\mathbf{*}_Q^0(\mathbf{M}) := \mathbf{Id}_Q$, $\mathbf{*}_Q^{i+1}(\mathbf{M}) := \mathbf{*}_Q^i(\mathbf{M}) \circ_Q (\mathbf{M} + \mathbf{Id}_Q)$ and $\mathbf{*}_Q^i(\mathbf{M}) := \bigcup_i \mathbf{*}_Q^i(\mathbf{M})$ see page 4.

Semilinear sets see Subsection 1.1.3

Petri net: $N = (P, T, W)$ with $W \in \mathbb{N}^{P \times T \cup T \times P}.$

Firing: $\mathbf{m}[t]\mathbf{m}'$, if $\mathbf{m} - W(\cdot, t) = \mathbf{m}' - W(t, \cdot) \in \mathbb{N}^P.$

Firing sequence $\mathbf{m}_0[w]\mathbf{m}_n$, if $\mathbf{m}_1, \dots, \mathbf{m}_{n-1}$ exist with $\mathbf{m}_0[t_1]\mathbf{m}_1[t_2]\dots[t_n]\mathbf{m}_n.$

Copies: $P^+ := \{p^+ \mid p \in P\}$, $P^- := \{p^- \mid p \in P\}$, $\mathbf{m}^- := \{p^- \mapsto \mathbf{m}(p) \mid p \in P\}$ and $\mathbf{m}^+ := \{p^+ \mapsto \mathbf{m}(p) \mid p \in P\}.$

Reachability relation for t : $\mathbf{R}(t) := \{\mathbf{m}^- + \mathbf{m}'^+ \mid \mathbf{m}[t]\mathbf{m}'\} = \mathbf{c}_t + \mathbf{Id}_P$ with $\mathbf{c}_t(p^-) := W(p, t)$ and $\mathbf{c}_t(p^+) := W(t, p)$ for all $p \in P$, $\mathbf{Id}_P := \mathbf{Id}_{\hat{P}}$ and $\hat{P} := \{(p^+, p^-) \mid p \in P\}.$

Reachability relation for T : $\mathbf{R}(T) := \bigcup_{t \in T} \mathbf{R}(t).$ concatenation: $\circ_P := \circ_{\hat{P}}.$

Reachability relation for N : $\mathbf{R}(N) := \mathbf{R}(T^*) := \mathbf{*}_P(\mathbf{R}(T))$ with $\mathbf{*}_P := \mathbf{*}_{\hat{P}}.$

Reachability problem: $(\mathbf{m}_0^- + \mathbf{m}_e^+) \in \mathbf{R}(N)?$ see page 7.

Petri nets with inhibitor arcs: $(P, T, W, I, \mathbf{m}_0, \mathbf{m}_e)$ with $W \in \mathbb{N}^{P \times T \cup T \times P}$ and $I \subseteq P \times T.$

Firing: $\mathbf{m}[t]\mathbf{m}'$, if $\mathbf{m} - W(\cdot, t) = \mathbf{m}' - W(t, \cdot) \in \mathbb{N}^P$ and $\forall p \in P (p, t) \in I \rightarrow \mathbf{m}(p) = 0.$ see page 9.

$\mathbf{R}(t_{p_1}(N')) := \{\mathbf{r} \in \mathbf{R}(N') \mid \mathbf{r}(p_1^-) = \mathbf{r}(p_1^+) = 0\}$, $\mathbf{R}(N) := \mathbf{*}_{P \setminus \{p_1\}}(\mathbf{R}(t_{p_1}(N')) \cup \mathbf{R}(\hat{t}))$ see page 10.

Expressions L_t, t, T, N, K_t and their carrier sets $C(e), \Gamma_t, P_T, A_t$ see page 12.

$t(N), t_{P'}(N)$ see proof of Lemma 3.1.

An expression T has the property \mathcal{T} if $\forall t \in T, \forall N_i = \ast_{P_{T_i}}(T_i) \in K_t$ the following conditions hold:

1. In recursive manner, T_i has

(a) the property \mathcal{T} , and

(b) For all $t' \in T_i$ it holds $\forall \mathbf{g} \in \{\mathbf{c}_{t'}\} \cup \Gamma_{t'} \exists w_{\mathbf{g}} \in C(t') \mathbf{g}(w_{\mathbf{g}}) = 1$,
 $\forall \mathbf{g}' \in \bigcup_{t' \in T_i} \{\mathbf{c}_{t'}\} \cup \Gamma_{t'} \setminus \{\mathbf{g}\} \mathbf{g}'(w_{\mathbf{g}}) = 0$.

2. $\forall \mathbf{g} \in \{\mathbf{c}_t\} \cup \Gamma_t, \forall p \in P_{T_i} \mathbf{g}(p^-) - \text{ind}(\mathbf{g})(p^-) = \mathbf{g}(p^+) - \text{ind}(\mathbf{g})(p^+)$, where

$$\text{ind}(\mathbf{g}) := \sum_{t' \in T_i, \mathbf{g}' \in \{\mathbf{c}_{t'}\} \cup \Gamma_{t'}} \mathbf{g}(w_{\mathbf{g}'}) \mathbf{g}'$$

3. $\forall w \in C(N_i) \setminus (P_{T_i}^+ \cup P_{T_i}^-) \sum_{g \in \Gamma_t} \mathbf{g}(w) > 0$.

4. There are multisets $\exists \mathbf{m}_+, \mathbf{m}_- \in \mathbf{R}(N_i)$ with $\forall p \in P_{T_i}$

$$\mathbf{m}_+ |_{P_{T_i}^-} \in (\mathbf{c}_t + \Gamma_t^*) |_{P_{T_i}^-} \wedge ((\forall \mathbf{g} \in \Gamma_t \mathbf{g}(p^-) = 0) \rightarrow \mathbf{m}_+(p^+) > \mathbf{m}_+(p^-)) \wedge$$

$$\mathbf{m}_- |_{P_{T_i}^+} \in (\mathbf{c}_t + \Gamma_t^*) |_{P_{T_i}^+} \wedge ((\forall \mathbf{g} \in \Gamma_t \mathbf{g}(p^+) = 0) \rightarrow \mathbf{m}_-(p^-) > \mathbf{m}_-(p^+)).$$

5. $\mathbf{c}_t |_{C(t)} \in \mathbf{R}(t)$.

See page 14 and Figure 2.

Order for multisets: Section 3.2 Size of an expression $S(T) = \sum_{t \in T} \{S(t) \mapsto 1\}$,

$S(t) := (S(K_t), b_2, b_5 + |\Gamma_t|)$, $b_i = 0$ if Condition $\mathcal{T}.i$ is fulfilled, and $b_i = 1$ otherwise. $S(K_t) = \sum_{N_i \in K_t} \{S(N_i) \mapsto 1\}$, $S(N_i) := (\mathbf{s}_m + \{|P_{T_i}| \mapsto 1\}, S(T_i), b_{1b}, |C(N_i)|)$

with $\mathbf{s}_m := \max\{\mathbf{s} \mid \exists \mathbf{g}, f, b_2, b'_{1b}, e, \mathbf{s}' \mathbf{s}'((\mathbf{s}, \mathbf{g}, b'_{1b}, f)) > 0, S(T_i)((\mathbf{s}', b_2, e)) > 0\}$.

$t|_L$ see proof of Lemma 3.3, $T \circ_Q T'$ see proof of Lemma 3.4.

Monotone transitive closure $\text{mTC}(\phi)$ see page 18.