Searching Paths of Constant Bandwidth

Bernd Borchert and Klaus Reinhardt

Universität Tübingen, Sand 13, 72076 Tübingen, Germany

{borchert,reinhard}@informatik.uni-tuebingen.de

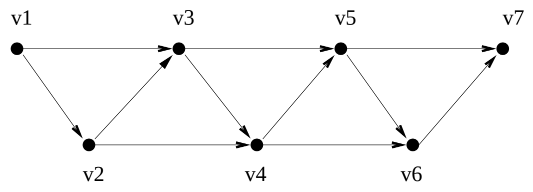
Abstract. As a generalization of paths, the notion of paths of bandwidth w is introduced. We show that, for constant $w \geq 1$, the corresponding search problem for such a path of length k in a given graph is NP-complete and fixed-parameter tractable (FPT) in the parameter k, like this is known for the special case $w = 1$, the LONGEST PATH problem. We state the FPT algorithm in terms of a guess and check protocol which uses witnesses of size polynomial in the parameter.

1 Introduction

A *path* within a graph is one of the most elementary notions of graph theory and its applications. The LONGEST PATH is the computational problem which asks for a given graph G and an integer k whether there is a path of length k in G which is simple, i.e. all vertices are different from each other. The LONGEST PATH is NP-complete [GJ79]. Moreover, the LONGEST PATH problem is fixed-parameter tractable (FPT) in the parameter k. This was shown by Monien [Mo85] and improved with respect to running time by Alon, Yuster, Zwick [AYZ95], using randomization techniques.

In this paper we generalize the notion of a path: a path of bandwidth w, or for short w-path, in a graph G is a sequence $(v_1, \ldots, v_n)$ of vertices such that for all v_i, v_j with $1 \leq j - i \leq w$ the pair (v_i, v_j) is an edge in G, see Fig. 1 for an example of a 2-path. 1-paths are paths in the usual sense. It will be easy to show that for every $w \geq 1$ the corresponding computational problem BANDWIDTH-w-PATH, which asks for a given graph G and an integer k whether there exists a simple w-path of length k in G, is NP-complete.

The BANDWIDTH-w-PATH problem for every w is fixed-parameter tractable in the parameter k, this will be shown according to the characterization of FPT $\cap$ NP by Cai, Chen, Downey & Fellows [CCDF95] via an “FPT guess and check protocol” using witnesses of size only dependent on the parameter. The runtime obtained for our guess and check protocol, for the case $w = 1$, which is the LONGEST PATH problem, and seen as a deterministic exhaustive search algorithm, is worse than the algorithms of Monien [Mo85] and Alon, Yuster, Zwick [AYZ95]. On the other hand, our algorithm is more easily stated and can immediately be applied to the BANDWIDTH-w-PATH problem. Moreover, the algorithms of [Mo85,AYZ95] do not seem to give better FPT guess and check protocols.



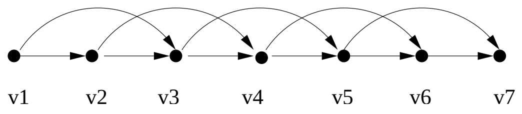


Fig. 1. Two drawings of the same 2-path of length 5, vertex-disjoint and deterministic

2 Paths of constant bandwidth

Let G be a digraph and let $w, k \geq 1$. A *path of bandwidth w and length k* in G is a sequence of $k+w$ vertices $(v_1, ..., v_{k+w})$ such that the pair (v_i, v_{i+j}) is an edge of G for every i with $1 \leq i \leq k$ and every j with $1 \leq j \leq w$. A path of bandwidth w and length k will also be called *w-path of length k* or, even shorter, *(w,k)-path*. A 1-path of length k is a path of length k in the usual sense. (For a path of length k some authors count the number of vertices while others count the number of edges – which is one less. In this paper we count the number of edges.) In Figures 2 and 3 some 2-paths and 3-paths are shown. Note that a $(w,1)$-path is a $(w+1)$-clique: every two nodes are connected by an edge. A (w,k)-path can actually be seen as a sequence of k $(w+1)$-cliques with two subsequent cliques "glued" together by their common w elements.

A (w,k)-path $(v_1, ..., v_{k+w})$ is *vertex-disjoint* if all v_i are different from each other, it is *simple* if all k w-tuples $(v_1, ..., v_w)$, $(v_2, ..., v_{w+1})$, $\ldots$, $(v_k, ..., v_{k+w})$ are different from each other. A vertex-disjoint (w,k)-path is simple, but not vice versa for $w \geq 2$, see Figure 3. A vertex-disjoint (w,k)-path, as a graph on its own, is the graph with $k+w$ vertices having bandwidth w and a maximal set of edges, that is why we choose the name "bandwidth" for the number w (see [PT99,GJ79] for the definition of bandwidth of a graph).

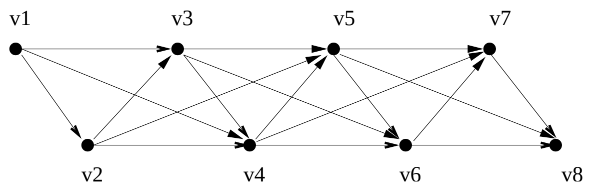


Fig. 2. A 3-path of length 5, vertex-disjoint and deterministic

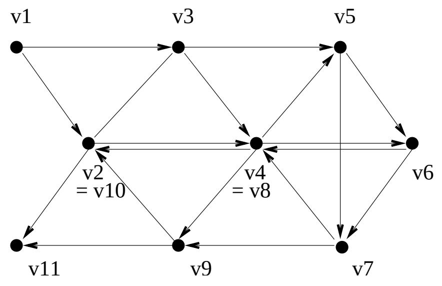


Fig. 3. A 2-path of length 10, deterministic and simple but not vertex-disjoint

Though the notion of w-paths within a graph G is a rather natural generalization of paths the authors could not find references for it in the literature. The closest concept found is the *w-ray* from Proskurowski & Telle [PT99], corresponding to a vertex-disjoint w-path (as a graph on its own).

A (w, k)-path $(v_1, ..., v_{k+w})$ is *deterministic in G* if for every $1 \leq i \leq k$ v_{i+w} is the only vertex in the graph G having the property that all edges $(v_i, v_{i+w}), \ldots, (v_{i+w-1}, v_{i+w})$ are edges of the graph. For example, a deterministic 1-path has the property that every vertex of it – besides the last one – has exactly one outgoing edge in G.

For $w < k$, a (w, k)-path $(v_1, ..., v_{k+w})$ is a *cycle of bandwidth w and length k*, for short w-cycle of length k or (w, k)-cycle, if $(v_{k+1}, \ldots, v_{k+w}) = (v_1, \ldots, v_w)$. The cycle is *vertex-disjoint* if $v_1, ..., v_k$ are different from each other, it is *simple* if $(v_1, ..., v_{k+w-1})$ is a simple w-path, see Fig. 4 for an example. For undirected graphs the definitions can be transfered literally. For a fixed w let BANDWIDTH-w-PATH be the set of pairs $\langle G, k\rangle$ such that the digraph G contains a simple (w, k)-path. BANDWIDTH-1-PATH = LONGEST-PATH. Let BANDWIDTH-PATH be the double-parameterized problem consisting of the triples $\langle G, w, k\rangle$ such that the digraph G contains a simple (w, k)-path.

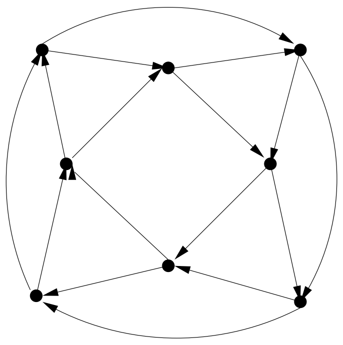


Fig. 4. A 2-cycle of length 8, deterministic and vertex-disjoint

Some variations of these problems: Let the prefixes UNDIRECTED- and DISJOINT- in front of these problem names indicate that the input graph is undirected, or, independently, that the path to be found has to be not only simple but vertex-disjoint, respectively. Let CYCLE instead of PATH in a problem name denote that the path to be found has to be a cycle. Call these further 7 problems the *variations* of the BANDWIDTH-w-PATH problem, resp. BANDWIDTH-PATH.

Proposition 1. *For every $w \geq 1$ the problem* BANDWIDTH-w-PATH *is* NP-*complete, likewise its variations.*

Proof. Obviously all problems are in NP. In order to show NP-completeness of BANDWIDTH-w-PATH we reduce LONGEST PATH to it. Let some directed graph $G = (V, E)$ be given. Let the vertices V' of the graph $\phi(G) = (V', E')$ consist of $3w$ copies v_i with $1 \leq i \leq 3w$ for each $v \in V$ and w copies $(u, v)_i$ with $1 \leq i \leq w$ for each $(u, v) \in E$ and let the edges be $E' = \{(u_i, u_j) \mid i < j, u \in V\} \cup \{(u_{i+w}, (u, v)_j), ((u, v)_i, v_{j+w}) \mid 1 \leq j \leq i \leq w, (u, v) \in E\} \cup \{((u, v)_i, (u, v)_j) \mid i < j, (u, v) \in E\}$.

It holds: G has a simple path of length k iff $\phi(G)$ has a simple w-path of length $(2k + 3)w$ iff $\phi(G)$ has a vertex-disjoint w-path of length $(2k + 3)w$. Observe that a w-path $(...u_{w+1}, ..., u_{2w}, (u, v)_1)$ in $\phi(G)$ can only be continued by $((u, v)_2, ...(u, v)_w, v_{w+1}, ...v_{2w}, ...)$ which forces a simple w-path to be vertex-disjoint and to correspond with a path in G. Starting with $(v_1, ...v_w, ...)$ or ending with $(...v_{2w+1}, ...v_{3w})$ allows to have the same length as starting in vertices corresponding to possibly unused edges in G. **q.e.d.**

DISJOINT-BANDWIDTH-PATH is NP-complete because LONGEST PATH is a subproblem and the length of a path is still at most linear. On the other hand, a simple path may have a length of $\binom{n}{w}$ and we conjecture PSPACE-completeness for BANDWIDTH-PATH.

We mention that for fixed w the problem of searching for a *deterministic* simple w-path of a given length k can be done in PTIME by a straightforward marking algorithm.

3 Fixed-Parameter Tractability

The following notion is from Downey & Fellows [DF92] though it can already be found – without giving it a name – in Monien [Mo85][p. 240, the two paragraphs before and after Th. 1, resp.].

Definition 1 (fixed-parameter tractability [Mo85,DF92]). *A computational problem consisting of pairs $\langle x, k\rangle$ is* fixed-parameter tractable *in the parameter k if there is a deciding algorithm for it having run-time $f(k) \cdot |x|^c$ for some recursive function f and some constant c.*

We use the following characterization of FPT $\cap$ NP by Cai, Chen, Downey & Fellows [CCDF95]:

Theorem 1 (Cai et al. [CCDF95]). *A language $L \in$ NP consisting of pairs $\langle x, k\rangle$ is fixed-parameter tractable in the parameter k iff there exists a recursive function $s(k)$ and a* PTIME *computable language C such that $\langle x, k\rangle \in L \iff \exists y \leq s(k):\ \langle x, k, y\rangle \in C$.*

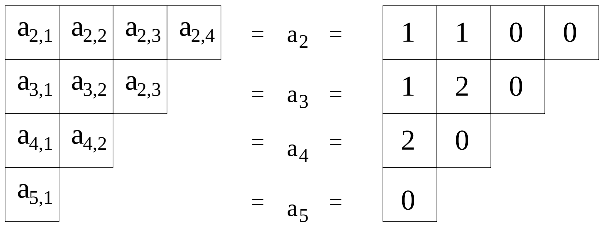
We call the function s the *witness size function*, and the language C the *witness checker*, and we say that these two together form an FPT *guess and check protocol* for L.

Theorem 2. *For every $w \geq 1$ the problem* BANDWIDTH-w-PATH *is fixed parameter tractable in the parameter k, likewise its variations. More specifically, there exists an* FPT *guess and check protocol for it with a witness size function $s(k) = \binom{k}{2} \cdot \log k$ and a witness checker having running time $O(w \cdot k^2 \cdot |E|^w \cdot |V|^w)$.*

Proof. We first consider the case $w = 1$, i.e. the LONGEST PATH problem. Afterwards we will see that the algorithm is generalizable to the BANDWIDTH-w-PATH problem for $w > 1$. We state an FPT guess and check protocol for LONGEST PATH with the witness size function $s(k) = \binom{k}{2} \cdot \log k$ and a witness checker with runtime $O(k^2 \cdot |E| \cdot |V|)$.

Let a digraph G with n vertices be given. We want to find out whether the graph contains a simple path $p = (v_1, \ldots, v_{k+1})$ of length k. We will work with *witnesses*. The intention of a witness is to tell the algorithm in the moment when it is trying to build an initial segment $(v_1, \ldots, v_i)$ of the simple path of length k which are the future vertices $v_{i+1}, \ldots, v_{k+1}$ of the simple path – so that the algorithm does not pick one of these future vertices as a part of the initial segment. Unfortunately, we cannot use the tuple $(v_1, \ldots, v_{k+1})$ as a witness, because that way we would have n^{k+1} potential witnesses, so we would need at least $(k+1)\log(n)$ bits to encode them, a number growing in n. But for the FPT guess and check protocol we need some witness size function $s(k)$ only dependent on k.

We choose the following kind of witnesses. A *witness* for such a simple path of length k consists of $k(k+1)/2 = \binom{k+1}{2}$ numbers $a_{i,j} \in \{0, 1, \ldots, k\}$, for $2 \leq i \leq k+1$ and $j \in \{1, \ldots, k-i+2\}$. The witness can be visualized as a half-matrix a, see Figure 5. Let a_i for $2 \leq i \leq k+1$ be the tuple $(a_{i,1}, \ldots, a_{i,k-i+2})$. We can restrict the witnesses to have these properties: a_i contains only numbers $\leq i-1$ and at least one 0. There is some redundancy, for example $a_{k+1,1}$ will



a2,1 a2,2 a2,3 a2,4 = a2 = 1 1 0 0
a3,1 a3,2 a2,3 = a3 = 1 2 0
a4,1 a4,2 = a4 = 2 0
a5,1 = a5 = 0

Fig. 5. Witness table for a simple path of length 4

always be 0. Nevertheless, the order of magnitude of the witness size function s(k) does not seem to be improvable by these “little savings”.

For every witness a the main algorithm C does the following: In each of the k steps $i = 2, 3, \ldots, k+1$ it uses a_i to compute for every vertex v a value $f_{a,i}(v)$, defined further below, which is either a vertex or has the value **nil** (standing for “not existing”), and stores this function for use in the following steps. The following pseudo code shows the main structure of the algorithm.

```
Main algorithm C
  Input: graph $G$, number $k \leq |G|$, and a witness $a$
  for every vertex $v$ set $f_{a,1}(v) := v$;
  for $i = 2, \ldots, k+1$ do
      for every vertex $v$ in $G$ do
          compute $f_{a,i}(v)$ and store it;
          if $i = k+1$ and $f_{a,i}(v) \neq$ nil ACCEPT and STOP;
  REJECT and STOP;
```

The computation of the value $f_{a,i}(v)$ – which is either **nil** or a vertex – is described in the pseudo code below. Assume w.l.o.g. that for each vertex there is a list of incoming edges (ending with the **nil** list element) in which the edges appear according to the order on the vertices. As a useful abbreviation let $f^d_{a,i}(v)$ for a vertex v and d with $1 \leq d \leq i+1$ be defined via $f^1_{a,i}(v) := v$, $f^2_{a,i}(v) := f_{a,i}(v)$, and $f^{d+1}_{a,i}(v) := f^d_{a,i-1}(f_{a,i}(v))$ with this value being **nil** in case $f_{a,i}(v)$ or $f^d_{a,i-1}(f_{a,i}(v))$ equals **nil**. Intuitively, $f^d_{a,i}(v)$ follows – starting in v – for growing $d = 1, \ldots, i+1$ the “backward path” given by the $f_{a,i-d}$-functions, see Figure 6. The upper index d numbers the vertices of this path, and the witness elements $a_{i,j} \geq 0$ will refer to this numbering. The information provided by $a_{i,j} \geq 0$ means that the $a_{i,j}$-th vertex in the “backward path” starting with a vertex u is the “reason” to dismiss the j-th attempt to assign a possible predecessor u of v to $f_{a,i}(v)$. (The set F in the following pseudo code collects such “reasons”.)

By easy induction on i, the following invariant will be guaranteed for every witness a, every i with $2 \leq i \leq k+1$, and every vertex v:

(Inv1) If $f_{a,i}(v) \neq$ **nil** then the “backward path” $(f^i_{a,i}(v), \ldots, f^2_{a,i}(v), f^1_{a,i}(v))$ is a simple path of length $i-1$.

```
Computing $f_{a,i}(v)$
  Input: $i$, $a$, and $v$. Already computed: $f_{a,1}, \ldots, f_{a,i-1}$.
  set $F := \{v\}$;
  set $j := 1$;
  if there are no incoming edges for $v$ set $f_{a,i}(v) :=$ nil and STOP;
  set $e = (u, v)$ to be the first edge incoming to $v$;
  while $e \neq$ nil do
        if $f_{a,i-1}(u) \neq$ nil and
          none of the vertices $f^1_{a,i-1}(u), f^2_{a,i-1}(u), \ldots, f^i_{a,i-1}(u)$ is in $F$ do
            set $c := a_{i,j}$;
            if $c = 0$
                set $f_{a,i}(v) := u$ and STOP;
              otherwise
                set $F := F \cup \{f^c_{a,i-1}(u)\}$;
                set $j := j + 1$;
        set $e = (u, v) :=$ next edge going into $v$;
  set $f_{a,i}(v) :=$ nil and STOP;
```

Verification of the main algorithm C: If the algorithm accepts, then it has found for this witness a a vertex v such that $f_{a,k+1}(v) \neq$ **nil**. By invariant (Inv1), case $i = k + 1$, the backward path starting in v is a simple path of length k.

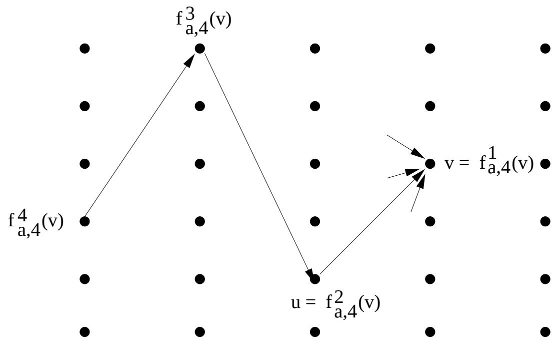


Fig. 6. A "backward path", starting in v

On the other hand assume that there is a simple path of length k in G. Let $s = (s_1, \dots, s_{k+1})$ be the lexicographically smallest among them (largest weight on s_{k+1}, unlike, for example, with decimal numbers). With the knowledge of this path and its vertices we will construct a witness b such that the main algorithm will accept for witness b.

```
Constructing $b$
  Input: $s_1, \dots, s_{k+1}$.
  for every vertex $v$ set $f_{b_1,1}(v) = v$;
  for $i = 2$ to $k + 1$ do
      set $e = (u, s_i) :=$ first edge going into $s_i$;
      set $F = \{s_i\}$;
      set $j := 1$;
      repeat
          while $f_{b_{i-1},i-1}(u) = \mathbf{nil}$
            or some of the vertices $f^1_{b_{i-1},i-1}(u), \dots, f^i_{b_{i-1},i-1}(u)$ is in $F$
              set $e = (u, s_i) :=$ next edge going into $s_i$;
          if there is a $c \in \{1, \dots, i\}$ such that $f^c_{b_{i-1},i-1}(u) \in \{s_{i+2}, \dots, s_{k+1}\}$
              set $b_{i,j} := c$ for the smallest such $c$;
              set $F := F \cup \{f^c_{b_{i-1},i-1}(u)\}$;
              set $j := j + 1$;
      until there is no such $c$;
      $b_{i,j} := 0$
      compute $f_{b_i,i}(v)$ for all vertices $v$;
```

The crucial invariant kept by this construction is the following:

(Inv2) For every i with $2 \le i \le k + 1$ it holds: $f_{b,i}(s_i) = s_{i-1}$.

The invariant holds via induction on i: the construction of b_i prevents $f_{b,i}(s_i)$ from choosing one of the vertices $s_{i+1}, \dots, s_{k+1}$ which will be needed in the future but which would be – without the witness – unknown at step i. Because there are at most $k - i + 1$ such vertices the repeat loop will always terminate and, moreover, the part b_i of the witness has sufficient size. For every $2 \le i \le k+1$ it is guaranteed that the computation of $f_{b,i}(s_i)$ will terminate, i.e. will be not-**nil**, because at least (s_{i-1}, s_i) is a suitable edge, and this will be the first suitable edge which $f_{b,i}(s_i)$ will find, i.e. $f_{b,i}(s_i) = s_{i-1}$, because otherwise $s = (s_1, \dots, s_{k+1})$ would not be lexicographically minimal. Invariant (Inv2) implies for $i = k+1$ that the backward path $(f^{k+1}_{b,k+1}(s_{k+1}), \dots, f^2_{b,k+1}(s_{k+1}), f^1_{b,k+1}(s_{k+1}))$ at s_{k+1} equals $s = (s_1, \dots, s_{k+1})$, i.e. the main algorithm C will accept the input graph for this witness b via a non-**nil** value of $f_{b,k+1}$ at vertex s_{k+1}. This finishes the correctness proof for the FPT guess and check protocol.

The running time of all $f_{a_i}(v)$ for a fixed i is $O(k \cdot |E|)$ as it is dominated by checking the backward path of length $\leq k$ for each edge incoming to v (we ignore some $\log(k)$ factors for the comparison algorithms). Therefore, the main algorithm C has runtime $O(k^2 \cdot |V| \cdot |E|)$. Representing all witnesses can be done with $\binom{k}{2} \cdot \log k$ bits, i.e. the witness size function can be chosen this way (note that the diagonal of the half matrix does not need to be stored – it can be assumed to consist of 0's). This finishes the proof that an FPT guess and check protocol exists for LONGEST PATH.

Cases $w > 1$. We first do a graph transformation. From the given graph G construct the following graph G': Consider all w-tuples $(v_1, \dots, v_w)$ of vertices of G. Make such a tuple a vertex of G' if the tuple represents a directed w-clique in G, i.e. (v_i, v_j) is an edge in G for $1 \leq i < j \leq w$. The edges in G' are defined to consist of the pairs of such w-cliques of the special form $((v_1, \dots, v_w), (v_2, \dots, v_w, v_{w+1}))$ such that also (v_1, v_{w+1}) is an edge in G. We have the property: G contains a simple w-path of length k iff G' contains a 1-path of length k. The witness checker consists therefore of this graph transformation and subsequently the checking algorithm C for $w = 1$ running on G'. In total the checking takes $O(w \cdot |V|^w \cdot |E|^w)$ time, the first w stems from a slightly higher comparison time for tuples. The witnesses size function does not change.

Variants: For the vertex disjoint case with $w > 1$ it is not enough to do the graph transformation, one has to go inside the checking algorithm C and maintain the vertex lists appropriately. **q.e.d.**

4 Conclusions and Open Questions

We introduced for every $w \geq 1$ the NP-complete problem BANDWIDTH-w-PATH and showed that it is fixed-parameter tractable in the length parameter k by presenting an FPT guess and check protocol for it, according to the characterization of Cai et al. [CCDF95].

As an open problem we suggest to study whether the witness size function, especially for the case LONGEST PATH, can be improved from the quasi-quadratic function $\binom{k}{2} \log k$ to some quasi-linear function, for example by the methods of Monien [Mo85] or Alon, Yuster & Zwick [AYZ95].

References

[AYZ95] N. Alon, R. Yuster, U. Zwick: *Color-Coding*, J. ACM 42(4): 844-856 (1995).

[CCDF95] L. Cai, J. Chen, R. G. Downey, M. R. Fellows: *On the Structure of Parameterized Problems in* NP, Inf. Comput. 123(1): 38-49 (1995).

[DF92] R. G. Downey, M. R. Fellows: *Fixed-Parameter Intractability*, Structure in Complexity Theory Conference 1992: 36-49.

[GJ79] M. R. Garey, D. S. Johnson: *Computers and Intractability*, Freeman, Ney York, 1979.

[Mo85] B. Monien: *How to find long paths efficiently*, Annals of Discrete Mathematics 25, 239-254 (1985).

[PT99] A. Proskurowski, J. A. Telle: *Classes of graphs with restricted interval models*, Discrete Mathematics & Theoretical Computer Science 3(4), 167-176 (1999).