Reachability in Petri nets
with Inhibitor arcs

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Overview

- Multisets, New operators $\circ_Q$ and $*_Q$ on multisets, Semilinearity
- Petri nets, Inhibitor arcs
- The reachability relation for Petri nets with one inhibitor arc
- Nested Petri Nets as normal form for expressions
- new Overview: Decision algorithm, Logic, Automata

Multisets

We write a multiset $\mathbf{f} \in \mathbb{N}^B$ as a set $\{b \mapsto f(b) \mid b \in B\}$, as a table $\begin{bmatrix} b_1 & b_2 & & b_n \\ f(b_1), & f(b_2), & \ldots, & f(b_n) \end{bmatrix}$

or as an n-ary vector $\begin{pmatrix} \mathbf{f}(b_1) \\ \mathbf{f}(b_2) \\ \vdots \\ \mathbf{f}(b_n) \end{pmatrix}$.

$$A \subseteq B \;\Rightarrow\; \mathbb{N}^A \subseteq \mathbb{N}^B$$

$$\mathbf{f} \in \mathbb{N}^A \;\wedge\; \mathbf{g} \in \mathbb{N}^B \;\Rightarrow (\mathbf{f}+\mathbf{g}) \in \mathbb{N}^{A \cup B}$$

$\emptyset$ with $\emptyset(x) = 0$ for all x is neutral element for +.

$$\mathbb{N}^A \cap \mathbb{N}^B = \mathbb{N}^{A \cap B}$$

$\text{sgn}(\mathbf{f}) := \{a \mid \mathbf{f}(a) > 0\}$, $\text{sgn}(\mathbf{M}) := \bigcup_{\mathbf{f} \in \mathbf{M}} \text{sgn}(\mathbf{f})$.

Restriction: $\mathbf{f}|_A := \{b \mapsto \mathbf{f}(b) \mid b \in A\}$ $\quad \mathbf{f}|_{\overline{A}} := \{b \mapsto \mathbf{f}(b) \mid b \notin A\}$, thus $\mathbf{f} = \mathbf{f}|_A + \mathbf{f}|_{\overline{A}}$.

A set $\mathbf{M} = \{\mathbf{m}_1, ..., \mathbf{m}_k\} \subseteq \mathbb{N}^A$ of multi-sets generate linear combinations:

$$\mathbf{M}^* := \{a_1\mathbf{m}_1 + ... + a_k\mathbf{m}_k | \forall i \leq k \ a_i \in \mathbb{N}\}$$

More generally, by $\mathbf{M}^0 := \{\emptyset\}$ and $\mathbf{M}^{i+1} := \mathbf{M}^i + \mathbf{M}$, we can define $\mathbf{M}^* := \bigcup_i \mathbf{M}^i$.

Linear set: $\mathbf{m}_c + \mathbf{M}^*$. Semilinear set: finite union of linear sets.

Semilinear sets: Smallest class of sets of multisets containing all finite sets of multisets and being closed under $\cup, +$ and $*$.

[GS65],[ES69]: The semilinear sets are also closed under $\cap$.

New operators $\circ_Q$ and $*_Q$ on multisets

For an unambiguous and injective binary relation Q and two sets of Multisets $\mathbf{M}$ and $\mathbf{N}$ we define

$$\mathbf{N}\circ_Q\mathbf{M} := \left\{ \mathbf{n}|_{\overline{\pi_1(Q)}} + \mathbf{m}|_{\overline{\pi_2(Q)}} \;\middle|\; \mathbf{n}\in\mathbf{N}, \mathbf{m}\in\mathbf{M}, \forall (a,b)\in Q\ \mathbf{n}(a) = \mathbf{m}(b) \right\}.$$

For example,

$$\left\{ \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} \right\} \circ_{\{(b_1,b_2)\}} \left\{ \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 8 \\ 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 7 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 5 \\ 5 \\ 5 \end{pmatrix} \right\}$$

or

$$\left\{ \begin{pmatrix} 3 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 5 \\ 2 \end{pmatrix} \right\} \circ_{\{(b_3,b_3)\}} \left\{ \begin{pmatrix} 8 \\ 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 7 \\ 2 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 2 \\ 3 \end{pmatrix} \right\} = \left\{ \begin{pmatrix} 11 \\ 9 \end{pmatrix}, \begin{pmatrix} 9 \\ 7 \end{pmatrix} \right\}$$

For $\pi_1(Q)$ and $\pi_2(Q)$ disjoint, we define $\mathbf{I}d_Q := \{\{a \mapsto 1, b \mapsto 1\} \mid (a,b) \in Q\}^*$ which is the neutral element for $\circ_Q$.

Obviously, it holds $\mathbf{N} \circ_\emptyset \mathbf{M} = \mathbf{N} + \mathbf{M}$ which makes $+$ with the neutral element $\mathbf{I}d_\emptyset = \{\emptyset\}$ a special case of the $\circ_Q$ operator.

Furthermore, for Q with $\pi_1(Q)$ and $\pi_2(Q)$ disjoint, we define $*_Q(\mathbf{M})$ as the closure of $\mathbf{M} \cup \mathbf{I}d_Q$ under $\circ_Q$ and the addition $\circ_\emptyset$.

In other words, $*_Q^0(\mathbf{M}) := \mathbf{I}d_Q$, $*_Q^{i+1}(\mathbf{M}) := *_Q^i(\mathbf{M}) \circ_Q \mathbf{M} + \mathbf{I}d_Q$ and $*_Q(\mathbf{M}) := \bigcup_i *_Q^i(\mathbf{M})$. Again, $*_\emptyset(\mathbf{M}) = \mathbf{M}^*$ is a special case.

Properties: $\mathbf{N} \circ_Q \mathbf{M} = \mathbf{M} \circ_{Q^{-1}} \mathbf{N}$

For $\mathbf{N}, \mathbf{M} \subseteq \mathbb{N}^A$ we get $\mathbf{N} \cap \mathbf{M} = \mathbf{N} \circ_{Q'} \mathbf{L} \circ_{Q''} \mathbf{M}$ with $Q' := \{(a, a') \mid a \in A\}$, $Q'' := \{(a'', a) \mid a \in A\}$ and $\mathbf{L} := \{\{a \mapsto 1, a' \mapsto 1, a'' \mapsto 1\} \mid a \in A\}^*$.

In general, $\mathbf{N} \circ_{Q'} \mathbf{L} \circ_{Q''} \mathbf{M}$ can only be written without brackets because $\pi_1(Q'') \cup (\mathrm{sgn}(\mathbf{M}) \setminus \pi_2(Q''))$ and $\pi_2(Q') \cup (\mathrm{sgn}(\mathbf{N}) \setminus \pi_1(Q'))$ are disjoint.

If, additionally, $\pi_2(Q'')$ and $\mathrm{sgn}(\mathbf{N})$ are disjoint and $\mathrm{sgn}(\mathbf{M})$ and $\pi_1(Q'))$ are disjoint, then $\mathbf{N} \circ_{Q'} \mathbf{L} \circ_{Q''} \mathbf{M} = \mathbf{L} \circ_{Q'^{-1} \cup Q''} (\mathbf{M} + \mathbf{N})$.

Semilinearity

$\circ_Q$ preserves semilinearity: Assume $\mathbf{N}$ and $\mathbf{M}$ are semilinear sets over A.
$\mathbf{N}'$ semilinear set over $A \setminus \pi_1(Q) \cup \pi_1(Q)'$.
$\mathbf{M}'$ semilinear set over $A \setminus \pi_2(Q) \cup \pi_2(Q)'$.

$\mathbf{E}'_Q := \{\{a' \mapsto 1, b' \mapsto 1\}, \{c \mapsto 1\} \mid (a,b) \in Q, c \in A\}^* = \{\mathbf{f} \mid \forall (a,b) \in Q\ \mathbf{f}(a') = \mathbf{f}(b')\}$
semilinear sets over the set $A \cup \pi_1(Q)' \cup \pi_1(Q)'$.

Thus, $\mathbf{N} \circ_Q \mathbf{M} = ((\mathbf{N}' + \mathbf{M}') \cap \mathbf{E}'_Q) \mid_{\overline{\pi_1(Q)' \cup \pi_1(Q)'}}$ is semilinear.

$*_Q$ does not preserve semilinearity:

Let $\mathbf{M} := \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}^*$, then $*_{\{(b_3, b_2)\}}(\mathbf{M}) = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| c \leq b 2^a \right\}$ not semilinear.

Petri net

We describe a *Petri net* as the triple $N=(P,T,W)$ with the places P, the transitions T and the weight function $W \in \mathbb{N}^{P\times T \cup T\times P}$. A transition $t \in T$ can fire from a marking $\mathbf{m} \in \mathbb{N}^P$ to a marking $\mathbf{m}' \in \mathbb{N}^P$, denoted by $\mathbf{m}[t\rangle\mathbf{m}'$, if

$$\mathbf{m} - W(.,t) = \mathbf{m}' - W(t,.) \in \mathbb{N}^P.$$

A *firing sequence* $w = t_1 ... t_n \in T^*$ can fire from $\mathbf{m}_0$ to $\mathbf{m}_n$, denoted by $\mathbf{m}_0[w\rangle\mathbf{m}_n$, if $\mathbf{m}_1, ... \mathbf{m}_{n-1}$ exist with $\mathbf{m}_0[t_1\rangle\mathbf{m}_1[t_2\rangle ... [t_n\rangle\mathbf{m}_n$.

Reachability problem: given net N with start- and end markings $\mathbf{m}_0, \mathbf{m}_e \in \mathbb{N}^P$, decide if there is a $w \in T^*$ with $\mathbf{m}_0[w\rangle\mathbf{m}_e$.

[May84][Kos84][Lam92]: decidable.

Let $P^+ := \{p^+ \mid p \in P\}$ and $P^- := \{p^- \mid p \in P\}$ be copies of the places and $\hat{P} := \{(p^+, p^-) \mid p \in P\}$. For $\mathbf{m}$ define the corresponding copies $\mathbf{m}^- := \{p^- \mapsto \mathbf{m}(p) \mid p \in P\}$ and $\mathbf{m}^+ := \{p^+ \mapsto \mathbf{m}(p) \mid p \in P\}$.

Reachability relation for a transition t:

$$\begin{aligned} \mathbf{R}(t) &:= \left\{ \mathbf{m}^- + \mathbf{m}'^+ \,\middle|\, \mathbf{m}[t\rangle\mathbf{m}' \right\} \\ &= \left\{ \mathbf{r} \in \mathbb{N}^{P^+ \cup P^-} \,\middle|\, \forall p \in P\ \mathbf{r}(p^-) - W(p,t) = \mathbf{r}(p^+) - W(t,p) \in \mathbb{N} \right\} \end{aligned}$$

Reachability relation for a set of transitions T as $\mathbf{R}(T) := \bigcup_{t \in T} \mathbf{R}(t)$.

Monotonicity: Whenever $\mathbf{m}[w\rangle\mathbf{m}'$, then also $(\mathbf{m}+\mathbf{n})[w\rangle(\mathbf{m}'+\mathbf{n})$ for any $\mathbf{n} \in \mathbb{N}^P$.

This corresponds to adding $\mathbf{I}d_P := \mathbf{I}d_{\hat{P}}$ and $\mathbf{R}(t) = \mathbf{c}_t + \mathbf{I}d_P$ is a linear set using $\mathbf{c}_t$ with $\mathbf{c}_t(p^-) = W(p,t)$ and $\mathbf{c}_t(p^+) = W(t,p)$ for all $p \in P$.

Concatenation of two firing sequences described by the operator $\circ_P := \circ_{\hat{P}}$

iteration described by $*_P := *_{\hat{P}}$.

The reachability relation of the petri net N is $\mathbf{R}(N) := \mathbf{R}(T^*) := *_P(\mathbf{R}(T))$.

The reachability problem: $(\mathbf{m}_0^- + \mathbf{m}_e^+) \in \mathbf{R}(N)$.

Corollary 1 *There is a firing sequence $w \in T^*$ with $\mathbf{m}_0[w\rangle\mathbf{m}_e$ in N if and only if*

$$\mathbf{m}_0^+ \circ_P \mathbf{R}(N) \circ_P \mathbf{m}_e^- = (\mathbf{m}_0^- + \mathbf{m}_e^+) \circ_A \mathbf{R}(N) = \{\emptyset\}$$

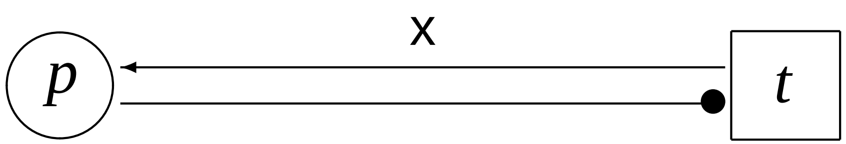
for $A := \{(p^-, p^-), (p^+, p^+) \mid p \in P\}$. In the other case $(\mathbf{m}_0^- + \mathbf{m}_e^+) \circ_A \mathbf{R}(N) = \emptyset$.

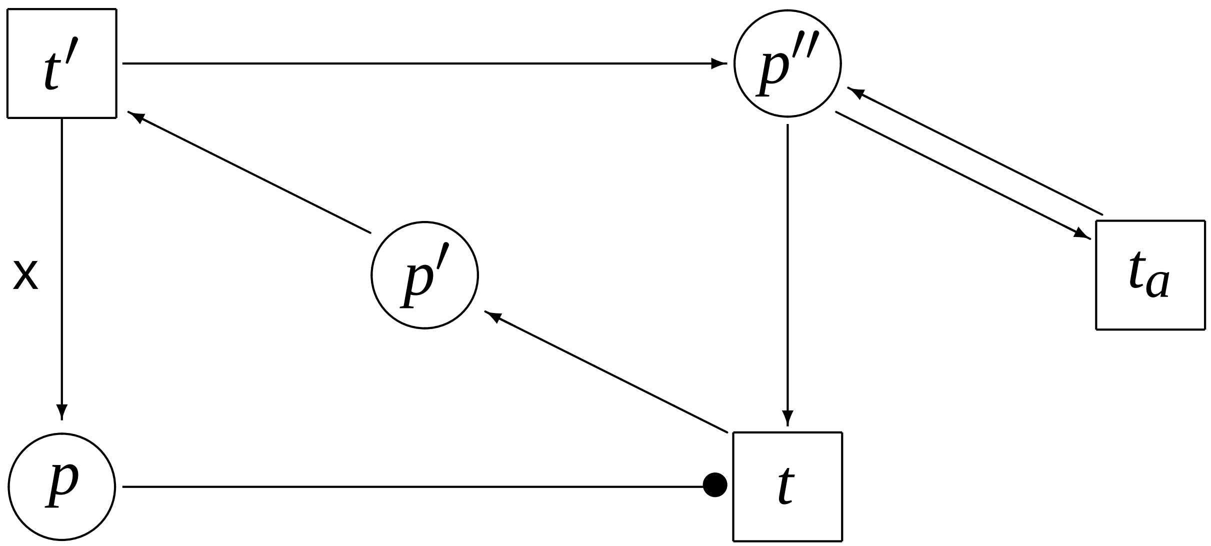
Inhibitor arcs

We describe such a Petri net as the 6-tuple $(P,T,W,I,\mathbf{m}_0,\mathbf{m}_e)$ with the places P, the transitions T, the weight function $W \in \mathbb{N}^{P\times T \cup T\times P}$, the inhibitor arcs $I \subseteq P \times T$ and, the start and end markings $\mathbf{m}_0, \mathbf{m}_e \in \mathbb{N}^P$. We will denote an inhibitor arc in the pictures by ———• .

$\mathbf{m}[t\rangle\mathbf{m}'$ only if $\forall p \in P\ (p,t) \in I \rightarrow \mathbf{m}(p) = 0$.

Lemma 1 *Each Petri net* $(P,T,W,I,\mathbf{m}_0,\mathbf{m}_e)$ *can be changed in such a way that the condition* $\forall p \in P, t \in T\ \ (p,t) \in I \rightarrow W(t,p) = 0$ *holds without changing the inhibitor arcs or the reachability problem.*





Lemma 2 *Each Petri net* $(P,T,W,I,\mathbf{m}_0,\mathbf{m}_e)$ *can be changed in a way such that the condition* $\forall p \in P, t \in T\ (p,t) \in I \rightarrow \mathbf{m}_0(p) = \mathbf{m}_e(p) = 0$ *holds by changing neither the inhibitor arcs, the condition in Lemma 1 nor the reachability problem.*

The reachability relation for Petri nets with one inhibitor arc

Given a Petri-net $(P, T, W, \{(p_1, \hat{t})\}, \mathbf{m}_0, \mathbf{m}_e)$. $w \in (T \setminus \{\hat{t}\})^*$.

$\mathbf{R}_1 = \mathbf{R}((P, T \setminus \{\hat{t}\}, W)) = *_P(\mathbf{R}(T \setminus \{\hat{t}\}))$

$\mathbf{R}_2 = \mathbf{R}_1 \cap \{\mathbf{r} \in \mathbb{N}^{P^-, P^+} \mid \mathbf{r}(p_1^-) = \mathbf{r}(p_1^+) = 0\}$

$\mathbf{R}_3 = \mathbf{R}_2 \cup \mathbf{R}(\hat{t})$

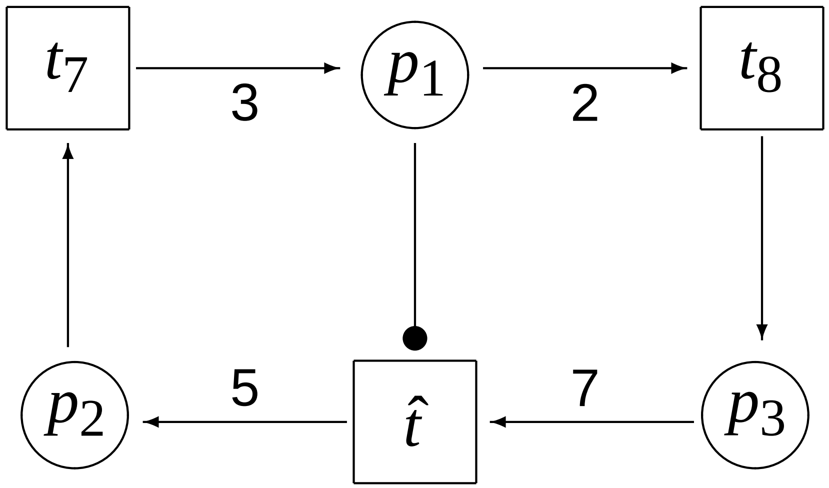
$\mathbf{R}_4 = *_{P \setminus \{p_1\}}(\mathbf{R}_3)$

Lemma 3 *Given a Petri-net* $(P,T,W,\{(p_1,\hat{t})\},\mathbf{m}_0,\mathbf{m}_e)$ *with only one inhibitor arc* $(p_1,\hat{t})$ *having the property of lemmata 1 and 2, then there is a firing sequence* $w \in T^*$ *with* $\mathbf{m}_0[w\rangle\mathbf{m}_e$ *if and only if*

$$\mathbf{m}_0^+ \circ_{P\setminus\{p_1\}} \mathbf{R}_4 \circ_{P\setminus\{p_1\}} \mathbf{m}_e^- = (\mathbf{m}_0^- + \mathbf{m}_e^+) \circ_A \mathbf{R}_4 = \{\emptyset\}$$

$A := \{(p^-,p^-),(p^+,p^+) \mid p \in P \setminus \{p_1\}\}$. *In the other case* $(\mathbf{m}_0^- + \mathbf{m}_e^+) \circ_A \mathbf{R}_4 = \emptyset$

Example:



with the start marking $\{p_2 \mapsto 4, p_3 \mapsto 2\}$ and the end marking $\{p_2 \mapsto 4, p_3 \mapsto 3\}$. We have $\mathbf{R}(t_7) = \{p_2^- \mapsto 1, p_1^+ \mapsto 3\} + \mathbf{I}d_P$, $\mathbf{R}(t_8) = \{p_1^- \mapsto 2, p_3^+ \mapsto 1\} + \mathbf{I}d_P$ and $\mathbf{R}(\hat{t}) = \{p_3^- \mapsto 7, p_2^+ \mapsto 5\} + \mathbf{I}d_{P\setminus\{p_1\}}$. This yields

$$\mathbf{R}_1 = \mathbf{R}((P, \{t_7, t_8\})) = *_P\left(\left\{\begin{bmatrix} p_2^- & p_1^+ \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} p_1^- & p_3^+ \\ 2 & 1 \end{bmatrix}\right\}\right) =$$

$$\left\{\begin{bmatrix} p_2^- & p_1^+ \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} p_1^- & p_3^+ \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} p_2^- & p_1^+ & p_3^+ \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} p_2^- & p_1^- & p_3^+ \\ 1 & 1 & 2 \end{bmatrix}, \begin{bmatrix} p_2^- & p_1^+ & p_3^+ \\ 2 & 2 & 2 \end{bmatrix}, \begin{bmatrix} p_2^- & p_3^+ \\ 2 & 3 \end{bmatrix}\right\}^* + \mathbf{I}d_P$$

and $\mathbf{R}_2 = \mathbf{R}_1 \circ_{\{(p_1^-,x),(p_1^+,y)\}} \{\emptyset\} = \left\{ \begin{bmatrix} p_2^- & p_3^+ \\ 2 & 3 \end{bmatrix} \right\}^* + \mathbf{I}d_{\{p_2,p_3\}}$.

We can cut the firing sequences in $(t_7 + t_8 + \hat{t})^* = ((t_7 + t_8)^* + \hat{t})^*$ into parts in $(t_7 + t_8)^*$ and $\hat{t}$ all starting and ending with no token on p_1. This yields $\mathbf{R}_3 = \mathbf{R}_2 \cup \mathbf{R}(\hat{t})$ and $\mathbf{R}_4 = *_{\{p_2,p_3\}}(\mathbf{R}_3) =$

$$\left\{ \begin{bmatrix} p_2^- & p_3^+ \\ 2 & 3 \end{bmatrix}, \begin{bmatrix} p_3^- & p_2^+ \\ 7 & 5 \end{bmatrix}, \begin{bmatrix} p_2^- & p_3^- & p_2^+ \\ 2 & 4 & 5 \end{bmatrix}, \begin{bmatrix} p_2^- & p_3^- & p_2^+ \\ 4 & 1 & 5 \end{bmatrix}, \begin{bmatrix} p_3^- & p_2^+ & p_3^+ \\ 7 & 3 & 3 \end{bmatrix}, \begin{bmatrix} p_3^- & p_2^+ & p_3^+ \\ 7 & 1 & 6 \end{bmatrix}, \ldots, \right.$$

$$\left. \begin{bmatrix} p_2^- & p_3^- & p_3^+ \\ 4 & 2 & 8 \end{bmatrix}, \begin{bmatrix} p_2^- & p_3^- & p_2^+ & p_3^+ \\ 5 & 1 & 1 & 7 \end{bmatrix}, \begin{bmatrix} p_3^- & p_2^+ & p_3^+ \\ 6 & 4 & 3 \end{bmatrix}, \begin{bmatrix} p_2^- & p_3^- & p_2^+ & p_3^+ \\ 4 & 2 & 4 & 3 \end{bmatrix} \right\}^* + \mathbf{I}d_{\{p_2,p_3\}}.$$

Nested Petri Nets as normal form for expressions

For every expression e, there is a *carrier set* $C(e) \supseteq \mathrm{sgn}(\mathbf{R}(e))\}$. $\mathbf{R}(e) \subseteq \mathbb{N}^{C(e)}$.

$\mathbf{R}$ is the evaluation function for an expression defined in a way such that is always commutes with the expression operators $*_P, \circ_Q, \cup$ and $+$, and the additional operator $\cap$.

Expression for an elementary transition: $t = L_t$ is an expression for the linear set $\mathbf{L}_t = \mathbf{R}(L_t) = \mathbf{c}_t + \Gamma_t^*$.

Example: $\Gamma_t = \{\{p^- \mapsto 1, p^+ \mapsto 1\} \mid p \in P\}$ leading to $\Gamma_t^* = \mathbf{I}d_P$. $C(t) := P^- \cup P^+ \cup sgn(\{c_t\} \cup \Gamma_t)$.

Expression for ***sets of transitions*****:** $T = t_1 \cup t_2 \ldots \cup t_l$ for expressions $t_i \in T$.

Expression for a ***sub-net***: $N = *_{P_T}(T)$ for N consisting of P_T and T.

Let $C(N) := C(T) := \bigcup_{t \in T} C(t)$.

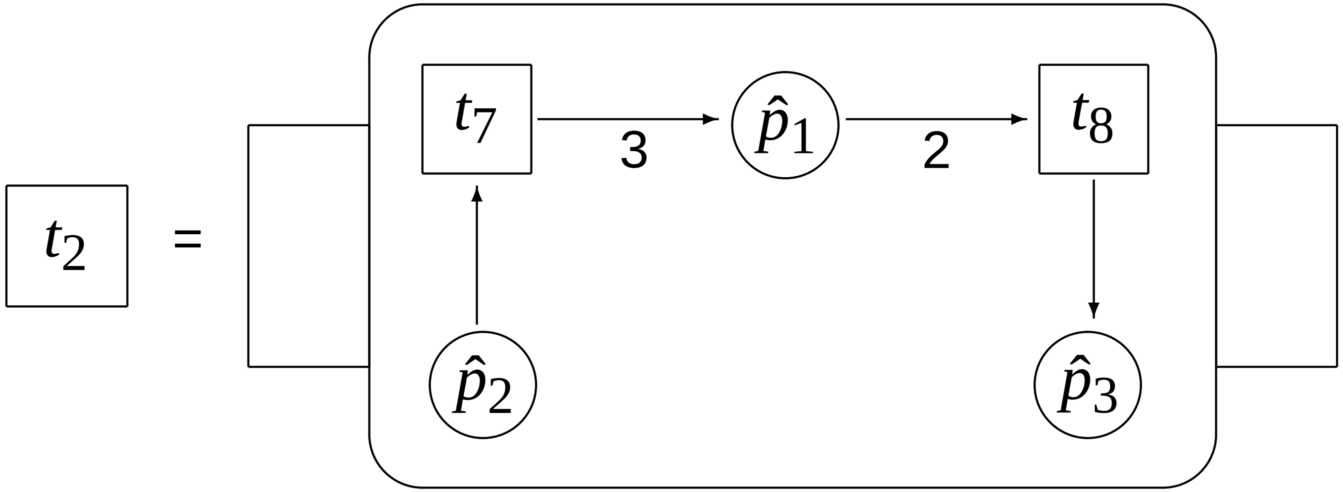
Expression for a ***generalized transition*****:** $t = L_t \circ_{Q_A} K_t$, where L_t again expresses a linear set, and K_t is a set of sub-nets and interpreted as expression $K_t = \sum_{N_i \in K_t} N_i$ where the $C(N_i)$ are pairwise disjoint.

Using $Q_A := \{(a,a) \mid a \in A\}$ with $A = \bigcup_{N_i \in K_t} C(N_i)$, we define $C(t) := \{a \mid (\mathbf{c}_t + \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g})(a) > 0\} \setminus A$. This means that the behavior of t is mainly described by the linear set $\mathbf{c}_t + \Gamma_t^*$ but it is additionally controlled by the reachability in the sub-nets N_i.

Example (continued): We identify $t_7 = \{\hat{p}_2^- \mapsto 1, \hat{p}_1^+ \mapsto 3\} + \mathbf{I}d_{\{\hat{p}_1,\hat{p}_2,\hat{p}_3\}}$, $t_8 = \{\hat{p}_1^- \mapsto 2, \hat{p}_3^+ \mapsto 1\} + \mathbf{I}d_{\{\hat{p}_1,\hat{p}_2,\hat{p}_3\}}$ and $\hat{t} = \{p_3^- \mapsto 7, p_2^+ \mapsto 5\} + \mathbf{I}d_{\{p_2,p_3\}}$. This yields the expressions $T_1 = t_7 \cup t_8$ and $N_1 = *_{\{\hat{p}_1,\hat{p}_2,\hat{p}_3\}}(T_1)$. On the next level, we get the generalized transition $t_2 =$

$$\left(\emptyset + \left\{ \begin{bmatrix} p_2^- & \hat{p}_2^- \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} p_3^- & \hat{p}_3^- \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} p_2^+ & \hat{p}_2^+ \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} p_3^+ & \hat{p}_3^+ \\ 1 & 1 \end{bmatrix} \right\}^* \right) \circ_{\{(\hat{p}_2^-,\hat{p}_2^-),(\hat{p}_3^-,\hat{p}_3^-),(\hat{p}_2^+,\hat{p}_2^+),(\hat{p}_3^+,\hat{p}_3^+)\}} N_1,$$

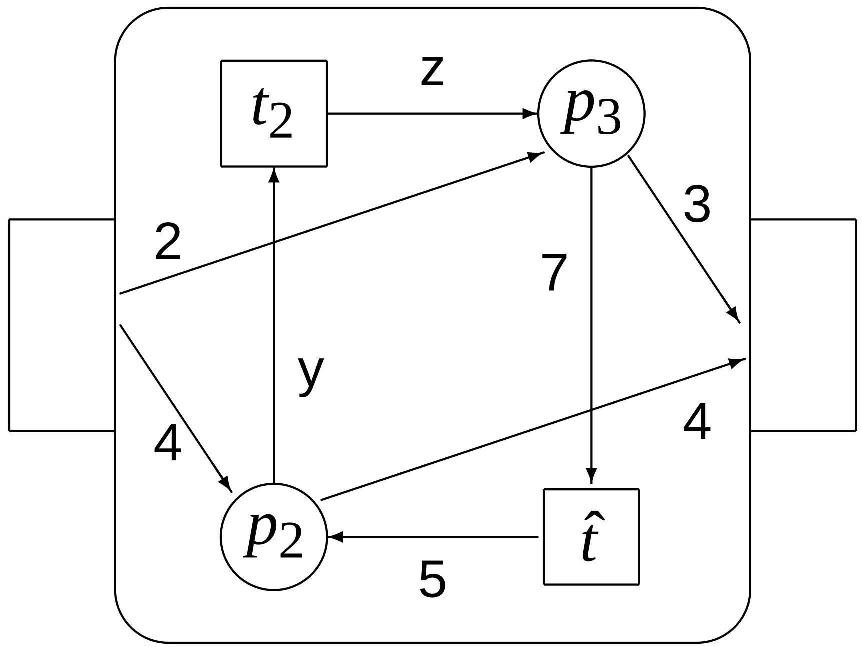
which we visualize as



$T_2 = t_2 \cup \hat{t}$ and $N_2 = \ast_{\{p_2,p_3\}}(T_2)$. On the top level, we get

$$T_3 = t_3 = \left[\frac{p_2^-}{4}, \frac{p_3^-}{2}, \frac{p_2^+}{4}, \frac{p_3^+}{3}\right] \circ_{\{(p_2^-,p_2^-),(p_3^-,p_3^-)\}} N_2,$$

which we visualize as



Expression		Carrier set
T		$C(T) = \bigcup$
	$\mathscr{T}.1.b$	
$t \in T$	$\{\mathbf{c}_t\} \cup \Gamma_t \subseteq \mathbb{N}$	$C(t) = P_T^- \cup P_T^+ \cup \{w_{\mathbf{c}_t}, w_{\mathbf{g}}, \ldots\}$
$N_i \in K_t \quad T_i$		$C(N_i) = C(T_i) = \bigcup$
	$\mathscr{T}.1.b$	
$t' \in T_i$	$\{\mathbf{c}_{t'}\} \cup \Gamma_{t'} \subseteq \mathbb{N}$	$C(t') = P_{T_i}^- \cup P_{T_i}^+ \cup \{w_{\mathbf{c}_{t'}}, w_{\mathbf{g}}', \ldots\}$
		$\ldots$

new Overview

- The property $\mathscr{T}$
- The size of an expression
- Additional operators working on expressions, Logic with mTC
- The main algorithm establishing property $\mathscr{T}$
- The reachability relation for Petri nets with inhibitor arcs
- Priority-Multicounter-Automata
- Restricted Priority- Multipushdown- Automata

The property $\mathscr{T}$

Definition 1 *An expression* T *has the property* $\mathscr{T}$ *if* $\forall t \in T, \forall N_i = *_{P_{T_i}}(T_i) \in K_t$ *the following 5 conditions hold:*

1. *In recursive manner,* T_i *has*

   (a) *the property* $\mathscr{T}$*, and*

   (b) *For all* $t' \in T_i$ *it holds* $\forall \mathbf{g} \in \{\mathbf{c}_{t'}\} \cup \Gamma_{t'}\ \exists w_{\mathbf{g}} \in C(t')\ \mathbf{g}(w_{\mathbf{g}}) = 1$, $\forall \mathbf{g}' \in \bigcup_{t' \in T_i} \{\mathbf{c}_{t'}\} \cup \Gamma_{t'} \setminus \{\mathbf{g}\}\ \ \mathbf{g}'(w_{\mathbf{g}}) = 0$.

2. $\forall \mathbf{g} \in \{\mathbf{c}_t\} \cup \Gamma_t, \forall p \in P_{T_i}\ \mathbf{g}(p^-) - ind(\mathbf{g})(p^-) = \mathbf{g}(p^+) - ind(\mathbf{g})(p^+)$, *where*

$$ind(\mathbf{g}) := \sum_{t' \in T_i, \mathbf{g}' \in \{\mathbf{c}_{t'}\} \cup \Gamma_{t'}} \mathbf{g}(w_{\mathbf{g}'})\mathbf{g}'$$

3. $\forall w \in C(N_i) \setminus (P^+_{T_i} \cup P^-_{T_i})\ \sum_{g \in \Gamma_t} \mathbf{g}(w) > 0.$

4. *There are multisets* $\exists \mathbf{m}_+, \mathbf{m}_- \in \mathbf{R}(N_i)$ *with* $\forall p \in P_{T_i}$

$$\mathbf{m}_+|_{P^-_{T_i}} \in (\mathbf{c}_t + \Gamma^*_t)|_{P^-_{T_i}} \wedge ((\forall \mathbf{g} \in \Gamma_t\ \mathbf{g}(p^-) = 0) \rightarrow \mathbf{m}_+(p^+) > \mathbf{m}_+(p^-)) \wedge$$

$$\mathbf{m}_-|_{P^+_{T_i}} \in (\mathbf{c}_t + \Gamma^*_t)|_{P^+_{T_i}} \wedge ((\forall \mathbf{g} \in \Gamma_t\ \mathbf{g}(p^+) = 0) \rightarrow \mathbf{m}_-(p^-) > \mathbf{m}_-(p^+)).$$

5. $\mathbf{c}_t\mid_{C(t)} \in \mathbf{R}(t)$.

Theorem 1 *For every expression T, we can effectively construct a T' with $\mathbf{R}(T) = \mathbf{R}(T')$ such that T' has property $\mathscr{T}$.*

Corollary 2 *The reachability problem for a Petri net with one inhibitor arc is decidable.*

The size of an expression

$\mathbf{m} :< \mathbf{m}'$ if there is an e with $\mathbf{m}(e) < \mathbf{m}'(e)$ and $\mathbf{m}(e') = \mathbf{m}'(e')$ for all $e' > e$.

[DM79]: Noetherian order on e's $\Rightarrow$ Noetherian order on $\mathbf{m}$'s.

$$S(T) := \sum_{t \in T} \{S(t) \mapsto 1\} .$$

$S(t) := (S(K_t), b_2, b_5 + |\Gamma_t|)$. Here, $b_i = 0$ if Condition $\mathscr{T}.i$ is fulfilled, and $b_i = 1$ otherwise.

$$S(K_t) = \sum_{N_i \in K_t} \{S(N_i) \mapsto 1\}.$$

$S(N_i) := (\mathbf{s}_m + \{|P_{T_i}| \mapsto 1\}, S(T_i), b_{1b}, |C(N_i)|)$ with
$\mathbf{s}_m := max\{\mathbf{s} \mid \exists \mathbf{s}' \ \mathbf{s}'((\mathbf{s}, ., ., .)) > 0, S(T_i)((\mathbf{s}', ., .)) > 0\}$.

Example (continued):
$S(t_7) = S(t_8) = (\emptyset, 0, 3)$, $S(T_1) = \{(\emptyset, 0, 3) \mapsto 2\}$,
$S(N_1) = (\{3 \mapsto 1\}, \{(\emptyset, 0, 3) \mapsto 2\}, 1, 6)$, $S(t_2) = (\{S(N_1) \mapsto 1\}, 1, 4)$,
$S(T_2) = \{S(t_2') \mapsto 1, (\emptyset, 0, 2) \mapsto 1\}$, $S(N_2) = (\{3 \mapsto 1, 2 \mapsto 1\}, S(T_2), 1, 4)$.

Lemma 4 *The ordering on* S *defined above is Noetherian*

Additional operators working on expressions

Lemma 5 *Let* $t = L_t \circ_Q K_t$ *be an expressions for a transition and* L *be (an expression for) a semi linear set. Then, we can construct an expression* $T' := t|_L$ *(with* $\mathbf{R}(T') = (\mathbf{R}(L_t) \cap \mathbf{R}(L)) \circ_Q \mathbf{R}(K_t)$*) where the occurring sizes* $S(t')$ *with* $t' \in T'$ *increase relatively to* $S(t)$ *only in the last position in the triple.*

Lemma 6 *Let* T *and* T' *be expressions for sets of transitions, and* Q *be a relation. Then, we can construct an expression* $T'' := T \circ_Q T'$ *(with* $\mathbf{R}(T'') = \mathbf{R}(T) \circ_Q \mathbf{R}(T')$*) where the occurring sizes* $S(t)$ *increase only in the last position in the triple and sum up in the first position.*

Lemma 7 *Let* N *be an expression for a subnet. Then, we can construct an equivalent expression for a transition* $t(N)$ *with* $\mathbf{R}(t(N)) = \mathbf{R}(N)$ *and* $t_{P'}(N)$ *with* $\mathbf{R}(t_{P'}(N)) = \{\mathbf{m} \in \mathbf{R}(N) \mid \forall p \in P' \; \mathbf{m}(p^-) = \mathbf{m}(p^+) = 0\}$.

Logic

Given a formula $\phi(x_1,...,x_k,x'_1,...,x'_k)$, then mTC$(\phi)$ denotes the smallest set $S \subset \mathbb{N}^{2k}$ containing all of the following:

- $(x_1,...,x_k,x_1,...,x_k)$ for $(x_1,...,x_k) \in \mathbb{N}^k$ (this stands for the identity),
- $(x_1,...,x_k,x'_1,...,x'_k)$ for $\phi(x_1,...,x_k,x'_1,...,x'_k)$
- $(x_1,...,x_k,x''_1,...,x''_k)$ for $(x_1,...,x_k,x'_1,...,x'_k),(x'_1,...,x'_k,x''_1,...,x''_k) \in S$, and
- $(x_1+x''_1,...,x_k+x''_k,x'_1+x''_1,...,x'_k+x''_k)$ for a $(x_1,...,x_k,x'_1,...,x'_k) \in S$ and $(x''_1,...,x''_k) \in \mathbb{N}^k$.

Corollary 3 *The emptiness and satisfiability is decidable for formulas with an FO+PLUS-formula inside and* $\wedge, \vee, \exists$ *and mTC operators outside.*

$\wedge$ corresponds to $\cap$ expressible with $\circ_Q$ Lemma 6

$\vee$ corresponds to $\cup$ expressible since T is already a union

$\exists$ remove the element

mTC is done by using Lemma 7.

The main algorithm establishing property $\mathscr{T}$

function reacheq(T):
begin
  repeat
    i:= 1
    while i$\leq$5 and $\forall t \in T, \forall N \in K_t$ Condition $\mathscr{T}$.i fulfilled
      do i:=i+1 **od**
    if i=6 **then return** T
      else T:=T' for T' according to treatment of Condition $\mathscr{T}$.i
  until i=6
end reacheq

in each step $S(T)$ decreases ($S(reacheq(T)) < S(T)$ if $T \neq reacheq(T)$); due to Lemma 4 the algorithm terminates.

Change of size $S(t)$ during the treatment of Condition $\mathscr{T}$.i:

	$S(t)$							
	$S(N_i)$							
		$S(t')$ for $t' \in T_i$						
	$\mathbf{s}_m + \{\lvert P_{T_i}\rvert \mapsto 1\}$	$S(K_{t'})$	b_2	$b_5 + \lvert\Gamma_{t'}\rvert$	b_{1b}	$\lvert C(N_i)\rvert$	b_2	$b_5 + \lvert\Gamma_t\rvert$
$\mathscr{T}$.1	-	-	-	-	$\downarrow$	$\uparrow$	$\uparrow$	-
$\mathscr{T}$.2	-	-	-	-	-	-	$\downarrow$	$\uparrow$
$\mathscr{T}$.3	-	-	-	-	-	$\downarrow$	$\uparrow$	$\uparrow$
$\mathscr{T}$.4	$\downarrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$
$\mathscr{T}$.5	-	-	-	-	-	-	-	$\downarrow$

Condition 1 Recursion and introducing witnesses

Let Condition 1 be not fulfilled by T_i; let $T_i' := reacheq(T_i)$, which terminates by induction since $S(T_i) < S(T)$. Construct T' from T by adding witnesses inside as for $T = T_2 = T_2 \cup \hat{t}$ in following continued example:

Replace t_7 and t_8 by $t_7' = \{\hat{p}_2^- \mapsto 1, \hat{p}_1^+ \mapsto 3, w_{\mathbf{c}_{t_7'}} \mapsto 1\} + \mathbf{Id}_{\hat{p}_1,\hat{p}_2,\hat{p}_3}$ and $t_8' = \{\hat{p}_1^- \mapsto 2, \hat{p}_3^+ \mapsto 1, w_{\mathbf{c}_{t_8'}} \mapsto 1\} + \mathbf{Id}_{\hat{p}_1,\hat{p}_2,\hat{p}_3} \;\Rightarrow\; T_1'' = t_7' \cup t_8'$ and $N_1'' = \ast_{\{\hat{p}_1,\hat{p}_2,\hat{p}_3\}}(T_1'')$.

$t_2' = (\emptyset + \{\{p_2^- \mapsto 1, \hat{p}_2^- \mapsto 1\}, \{p_3^- \mapsto 1, \hat{p}_3^- \mapsto 1\}, \{p_2^+ \mapsto 1, \hat{p}_2^+ \mapsto 1\}, \{p_3^+ \mapsto 1, \hat{p}_3^+ \mapsto 1\}, \{w_{c_{t_7'}} \mapsto 1\}, \{w_{c_{t_8'}} \mapsto 1\}\}^*), \circ_{\{(\hat{p}_2^-,\hat{p}_2^-),(\hat{p}_3^-,\hat{p}_3^-),(\hat{p}_2^+,\hat{p}_2^+),(\hat{p}_3^+,\hat{p}_3^+)\}} N_1''$ and $T_2' = t_2' \cup \hat{t}$.

The new sizes are now $S(t_7') = S(t_8') = (\emptyset, 0, 3) = S(t_7)$,
$S(T_1'') = \{(\emptyset, 0, 3) \mapsto 2\} = S(T_1)$,
$S(N_1'') = (\{3 \mapsto 1\}, \{(\emptyset, 0, 3) \mapsto 2\}, 0, 8) < S(N_1)$,
$S(t_2') = (\{S(N_1'') \mapsto 1\}, 1, 6) < S(t_2)$,
$S(T_2') = \{S(t_2') \mapsto 1, (\emptyset, 0, 2) \mapsto 1\} < S(T_2)$.

Condition 2 Quantitative consistency

Let Condition 2 be not fulfilled by T_i. The set $\mathbf{L} :=$

$$\left\{ \mathbf{g} \in \mathbb{N}^{C_{\mathbf{L}}} \,\middle|\, \forall p \in \bigcup_{N_i \in K_t} P_{T_i}\ \mathbf{g}(p^-) - ind(\mathbf{g})(p^-) = \mathbf{g}(p^+) - ind(\mathbf{g})(p^+) \right\}$$

is a Presburger set. Construct $T' := T \setminus \{t\} \cup t|_L$ using Lemma 5. In the example L is characterized by the following three equations:

$2\mathbf{g}(w_{\mathbf{c}_{t'_8}}) = 3\mathbf{g}(w_{\mathbf{c}_{t'_7}})$, $\mathbf{g}(\hat{p}_2^-) - \mathbf{g}(w_{\mathbf{c}_{t'_7}}) = \mathbf{g}(\hat{p}_2^+)$, $\mathbf{g}(\hat{p}_3^-) = \mathbf{g}(\hat{p}_3^+) - \mathbf{g}(w_{\mathbf{c}_{t'_8}})$.

Their solutions are described by the linear set $L_{t''_2} = L_{t'_2} \cap L =$

$$\emptyset + \left\{ \begin{bmatrix} \frac{p_2^-}{1}, \frac{\hat{p}_2^-}{1}, \frac{p_2^+}{1}, \frac{\hat{p}_2^+}{1} \end{bmatrix}, \begin{bmatrix} \frac{p_3^-}{1}, \frac{\hat{p}_3^-}{1}, \frac{p_3^+}{1}, \frac{\hat{p}_3^+}{1} \end{bmatrix}, \begin{bmatrix} \frac{w_{\mathbf{c}_{t'_7}}}{2}, \frac{w_{\mathbf{c}_{t'_8}}}{3}, \frac{p_2^-}{2}, \frac{\hat{p}_2^-}{2}, \frac{p_3^-}{3}, \frac{\hat{p}_3^-}{3} \end{bmatrix} \right\}^*$$

$\Rightarrow t_2'' = L_{t_2''} \circ_{\{(a,a)|a\in\{\hat{p}_2^-,\hat{p}_2^+,\hat{p}_3^-,\hat{p}_3^+\}\}} N_1''$ with $S(t_2'') = (\{S(N_1'') \mapsto 1\}, 0, 3) < S(t_2')$.

Adding the witnesses leads to $L_{t_2'''} =$

$$\emptyset + \left\{ \begin{bmatrix} p_2^- & \hat{p}_2^- & p_2^+ & \hat{p}_2^+ & w_1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} p_3^- & \hat{p}_3^- & p_3^+ & \hat{p}_3^+ & w_2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} w_{\mathbf{c}_{t_7'}} & w_{\mathbf{c}_{t_8'}} & p_2^- & \hat{p}_2^- & p_3^- & \hat{p}_3^- & w_3 \\ 2 & 3 & 2 & 2 & 3 & 3 & 1 \end{bmatrix} \right\}^*$$

(we omit the witness for $\emptyset$.) with $S(t_2''') = S(t_2'') = (\{S(N_1'') \mapsto 1\}, 0, 3)$.

Defining $T_2''' = t_2''' \cup \hat{t}'$ with $S(T_2''') = S(T_2'')$ and $N_2''' = *_{\{\hat{p}_1,\hat{p}_2,\hat{p}_3\}}(T_2''')$ with $S(N_2''') = (\{3 \mapsto 1, 2 \mapsto 1\}, S(T_2'''), 0, 8) < S(N_2'') = (\{3 \mapsto 1, 2 \mapsto 1\}, S(T_2'''), 1, 4)$ we get

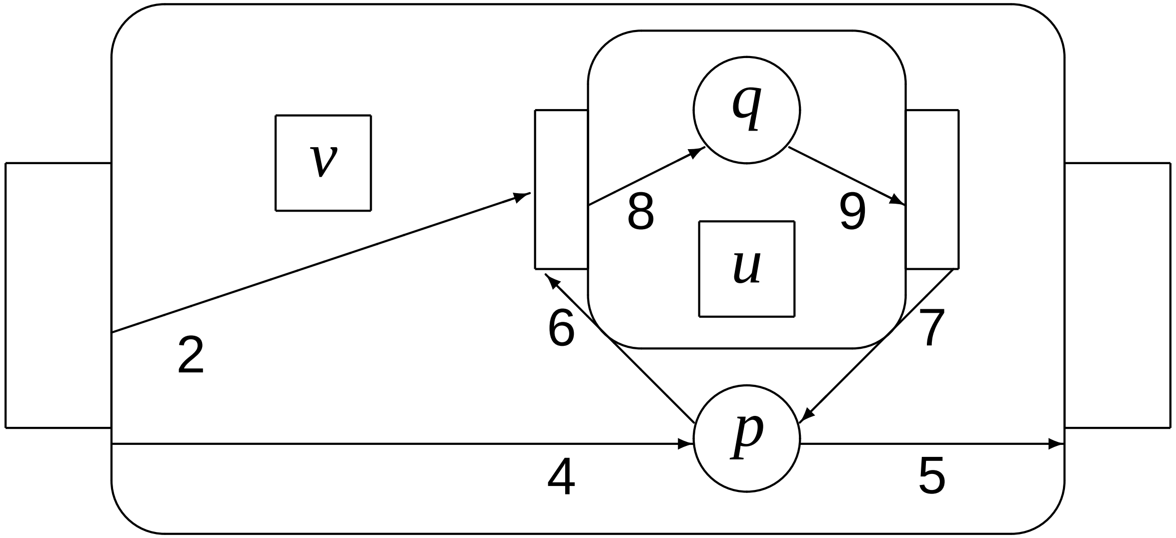
$$t_3' = \left(\begin{bmatrix} p_2^- & p_3^- & p_2^+ & p_3^+ \\ 4 & 2 & 4 & 3 \end{bmatrix} + \left\{ \begin{bmatrix} w_1 \\ 1 \end{bmatrix}, \begin{bmatrix} w_2 \\ 1 \end{bmatrix}, \begin{bmatrix} w_3 \\ 1 \end{bmatrix}, \begin{bmatrix} w_{\mathbf{c}_{\hat{t}'}} \\ 1 \end{bmatrix} \right\}^* \right) \circ_{\{(a,a)|a\in\{p_2^-,p_2^+,p_3^-,p_3^+\}\}} N_2'''.$$

Condition 3 Elimination of witnesses

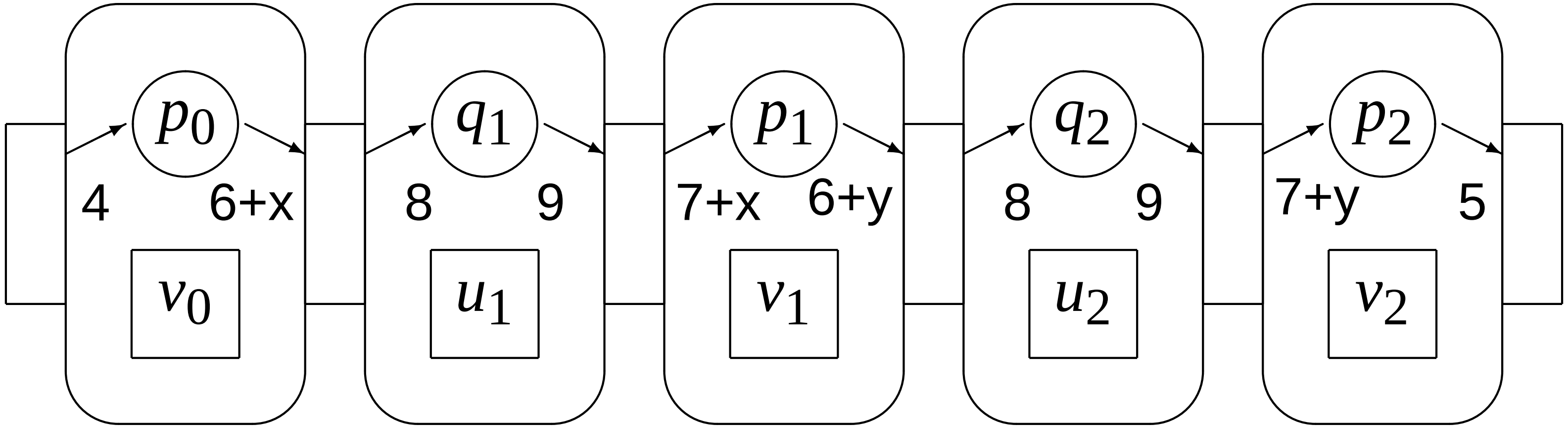
Let Condition 3 be not fulfilled by witness $w \in C(N_i) \setminus (P^+_{T_i} \cup P^-_{T_i})$. Replace N_i by some expression $\hat{T}$ with $\mathbf{R}(\hat{T}) = \mathbf{R}(N_i) \circ_{(w,w)} \mathbf{c}_t|_w$ since for all $\mathbf{m} \in \mathbf{L}_t$, we have $\mathbf{m}(w) = \mathbf{c}_t(w)$. Then, we can replace in

$$T' := T \setminus \{t\} \cup (L_t|_{\overline{\{w\}}} \circ_{Q \setminus Q_{C(N_i)}} (K_t \setminus \{N_i\})) \circ_{Q_{C(N_i) \setminus \{w\}}} \hat{T}$$

Example: Consider t with $\mathbf{c}_t = \begin{bmatrix} w & p^- & p^+ \\ 2, & 4, & 5 \end{bmatrix}$, $\forall \mathbf{g} \in \Gamma_t\ \mathbf{g}(w) = 0$, $K_t = \{*_{\{p\}}(v \cup t_j)\}$, and $\mathbf{c}_{t_j} = \begin{bmatrix} w & p^- & p^+ & q^- & q^+ \\ 1, & 6, & 7, & 8, & 9 \end{bmatrix}$, $K_{t_j} = \{*_{\{q\}}(u)\}$.



Then t' is defined such that $\mathbf{c}_{t'} = \left[\frac{p_0^-}{4}, \frac{p_0^+}{6}, \frac{q_1^-}{8}, \frac{q_1^+}{9}, \frac{p_1^-}{7}, \frac{p_1^+}{6}, \frac{q_2^-}{8}, \frac{q_2^+}{9}, \frac{p_2^-}{7}, \frac{p_2^+}{5} \right]$, furthermore, $\left[\frac{p_1^-}{1}, \frac{p_0^+}{1} \right], \left[\frac{p_2^-}{1}, \frac{p_1^+}{1} \right] \in \Gamma_{t'}$ and $K_t = \{ *_{\{p_0\}}(v_0), *_{\{q_1\}}(u_1), *_{\{p_1\}}(v_1), *_{\{q_2\}}(u_2), *_{\{p_2\}}(v_2) \}$, where p_i, q_i, v_i and u_i are replacements caused by disjointness condition in Lemma 6.



The variables x and y illustrate the effect of the periods in $\Gamma_{t'}$ which originate from the (omitted) periods of t_j.

Condition 4 Elimination of bounded places

Condition 4 is decidable by two *covering graph* constructions for every i: Every node in the covering graph $CG_{(i,+)}$ ($CG_{(i,-)}$, respectively) has a marking from $(\mathbb{N}\cup\{\omega\})^{P^-_{T_i}}$ ($(\mathbb{N}\cup\{\omega\})^{P^+_{T_i}}$, respectively). The root of the covering graph $CG_{(i,+)}$ has the marking $\mathbf{c}_t|_{P^-_{T_i}} + \omega^{\{p^- | \exists \mathbf{g} \in \Gamma \mathbf{g}(p^-)>0\}}$.

For a node in $CG_{(i,+)}$ marked with $\mathbf{m}$, we construct T_i' with $\mathbf{R}(T_i') = \{\mathbf{g} \in \mathbf{R}(T_i) \mid \mathbf{g}|_{P^-_{T_i}} \leq \mathbf{m}\}$ using Lemma 5 as $T_i' := \{t'|_{\{\mathbf{g}\in \mathbf{L}_{t'} |\ \mathbf{g}|_{P^-_{T_i}} \leq \mathbf{m}\}} \mid t' \in T_i\}$. Compute $T_i'' := reacheq(T_i')$ recursively.

For every $t'' \in T_i''$, add to the covering graph $CG_{(i,+)}$ a new node

$$\mathbf{m}' := \mathbf{m} - \mathbf{c}_{t''}|_{P^-_{T_i}} + \{p^- \mapsto (\mathbf{c}_{t''}(p^+) + \omega \sum_{\mathbf{g}\in\Gamma_{t''}} (p^+)) \mid p \in P_{T_i}\}$$

If $\mathbf{m}' > \mathbf{m}''$ for an $\mathbf{m}''$ on the path from the root to $\mathbf{m}$, then we set $\mathbf{m}' := \mathbf{m}' + \omega(\mathbf{m}' - \mathbf{m}'')$.

If for all i a node marked with $\omega^{P^-_{T_i}}$ is in $CG_{(i,+)}$ and, analogously, a node marked with $\omega^{P^+_{T_i}}$ is in $CG_{(i,-)}$, then the Condition 4 is fulfilled. Otherwise, we calculate

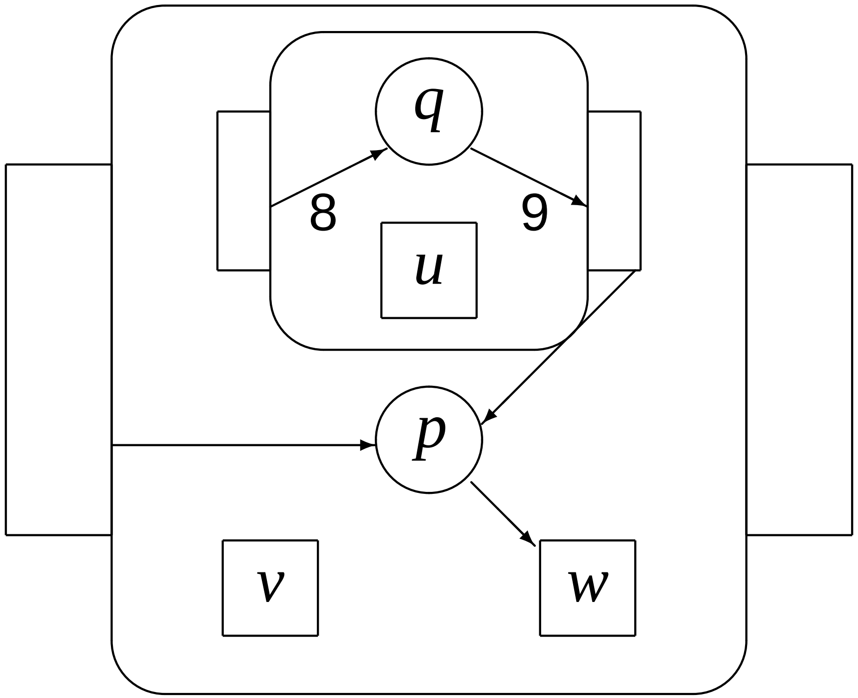
$$k := \min_{\sigma \in \{+,-\}} \max_{path \subseteq CG_{(i,\sigma)}} \min_{p \in P_{T_i}} \max_{\mathbf{m} \in path} \mathbf{m}(p^\sigma)$$

and construct $T' := T \setminus \{t\} \cup \bigcup_{p \in P_{T_i}} U(p)$ (restrict place p to k).

$$U(p) = (L_t \mid_{\overline{\{p^-,p^+\}}} \circ_{Q \setminus Q_{C(N_i)}} (K_t \setminus \{N_i\})) \circ_{Q_{C(N_i) \setminus \{p^-,p^+\}}} T^k_{\mathbf{c}_t(p^-),\mathbf{c}_t(p^+)}$$

$T^{l-1}_{j,h}$ describes firing sequences in N_i with the following property: They start with a marking $\mathbf{m}_0$ with $\mathbf{m}_0(p) = j$, end with a marking $\mathbf{m}_1$ with $\mathbf{m}_1(p) = h$, and meanwhile the number tokens on p is always less than l.

Example: Let $t = (\mathbf{c} + \Gamma^*) \circ_{\{p\} \cup P} \ast_{\{p\} \cup P} (N_i\})$ with $\mathbf{c}(p^-) = 1$, $N_i = v \cup w \cup t_j$ and $t_j = (\mathbf{c}_j + \Gamma_j^*) \circ_{\{q\} \cup Q} \ast_{\{q\} \cup Q} (u)$ with $\mathbf{c}_j(p^+) = 1$, $\mathbf{c}_j(q^-) = 8$ and $\mathbf{c}_j(q^+) = 9$

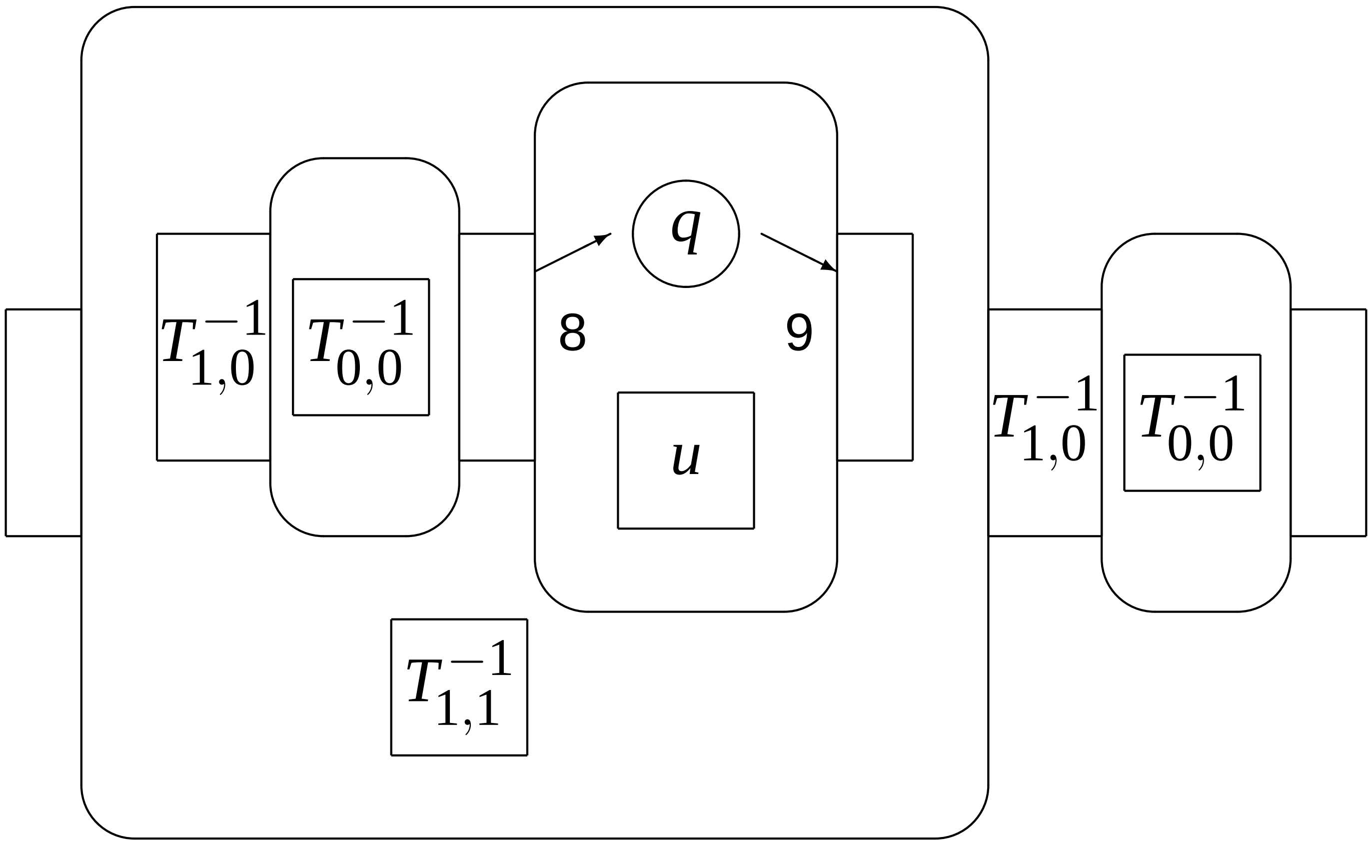


$k = 1 \;\Rightarrow\;$ firing sequences are restricted to the regular expression $((wv^*t_j) + v)^*wv^*$.

$T_{0,0}^{-1}$ and $T_{1,1}^{-1}$ only consist of a copy of v, $T_{1,0}^{-1}$ only of a copy of w and $T_{0,1}^{-1}$ only of a copy of t_j.

$T_{0,0}^{0} = t(\ast_P(T_{0,0}^{-1}))$ correponds to v^*, $T_{1,1}^{0} = T_{1,0}^{-1} \circ_P T_{0,0}^{0} \circ_P T_{0,1}^{-1} \cup T_{1,1}^{-1}$ corresponds to $wv^*t_j + v$ and $T_{1,0}^{1} = T_{1,1}^{1} \circ_P T_{1,0}^{0} = t(\ast_P(T_{1,1}^{0})) \circ_P T_{1,0}^{-1} \circ_P T_{0,0}^{0}$

Every t' in $(\mathbf{c} + \Gamma^*)|_{\overline{\{p^-,p^+\}}} \circ_{Q_{C(N_i)\setminus\{p^-,p^+\}}} T_{1,0}^{1}$ looks like



$T_{1,0}^{-1}$
$T_{0,0}^{-1}$
q
8
9
u
$T_{1,0}^{-1}$
$T_{0,0}^{-1}$
$T_{1,1}^{-1}$

Corollary 4 *If the conditions 1 - 4 hold for t, then it holds*

$$\forall \mathbf{f} \in \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g} + \Gamma_t^* \exists k \geq 2 \ (\mathbf{c}_t + k\mathbf{f}) \mid_{C(t)} \in \mathbf{R}(t)$$

This immediately follows from:

Lemma 8 *If the conditions 1 - 4 hold for t, then it holds*

$$\forall \mathbf{f} \in \sum_{\mathbf{g} \in \Gamma_t} \mathbf{g} + \Gamma_t^* \forall \mathbf{e} \in (\Gamma_t \cup -\Gamma_t)^* \exists k \geq 2 \ \left\{ (\mathbf{c}_t + k\mathbf{f}) \mid_{C(t)}, (\mathbf{c}_t + k\mathbf{f} + \mathbf{e}) \mid_{C(t)} \right\} \subseteq \mathbf{R}(t)$$

Condition 5 Making the constant firing

If Condition 5 is not fulfilled for t then, according to Corollary 4, for $\mathbf{f} = \sum_{\mathbf{g} \in \Gamma} \mathbf{g}$, there exists a (smallest) k such that $(c + k\mathbf{f})|_{C(t)} \in \mathbf{R}(t)$. So we decompose L_t such that $\mathbf{R}(L_t) = \mathbf{R}(L_t + k\mathbf{f}) \cup \bigcup_{\mathbf{g} \in \Gamma} \bigcup_{j \leq k} \mathbf{R}(\mathbf{c}_t + j\mathbf{g} + (\Gamma_t \setminus \{\mathbf{g}\})^*)$. Set

$$
\begin{aligned}
T' := T \setminus \{t\} \quad & \cup \{t' \mid K_{t'} = K_t, \Gamma'_t = \Gamma_t \wedge \mathbf{c}_{t'} = \mathbf{c}_t + k\mathbf{f})\} \\
& \cup \{t' \mid \exists j \leq k, \mathbf{g} \in \Gamma \; \Gamma'_t = \Gamma_t \setminus \{\mathbf{g}\}) \wedge \mathbf{c}_{t'} = \mathbf{c}_t + j\mathbf{g})\}.
\end{aligned}
$$

The size $S(t')$ is smaller than $S(t)$ since b_5 is now zero or $|\Gamma \setminus \{\mathbf{g}\}| < |\Gamma|$.

The reachability relation for Petri nets with inhibitor arcs

Theorem 2 *In a Petri-net* $(P,T,W,I,\mathbf{m}_0,\mathbf{m}_e)$ *with*

$$\exists \mathbf{g} \in \mathbb{N}_+^P \; \forall p,p' \in P \; \mathbf{g}(p) \leq \mathbf{g}(p') \rightarrow (\forall t \in T \; (p',t) \in I \rightarrow (p,t) \in I),$$

we can construct an expression T_g *such that there is a firing sequence* $w \in T^*$ *with* $\mathbf{m}_0[w\rangle\mathbf{m}_e$ *if and only if* $\mathbf{R}(T_g)$ *is (*$= \{\emptyset\}$ *and) not empty.*

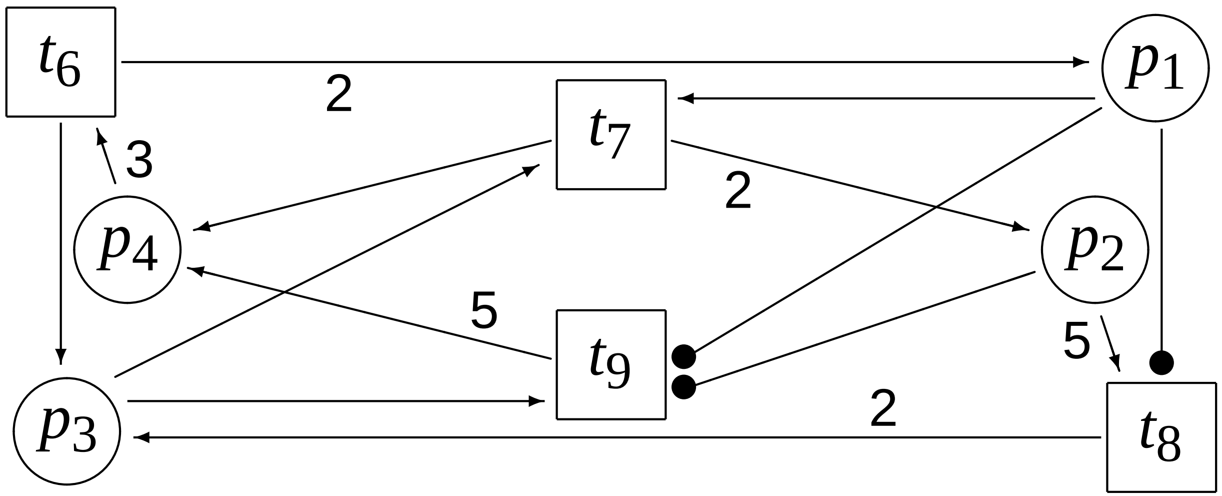
With Theorem 1 we derive the following:

Corollary 5 *The reachability problem for a Petri net* $(P,T,W,I,\mathbf{m}_0,\mathbf{m}_e)$ *with*

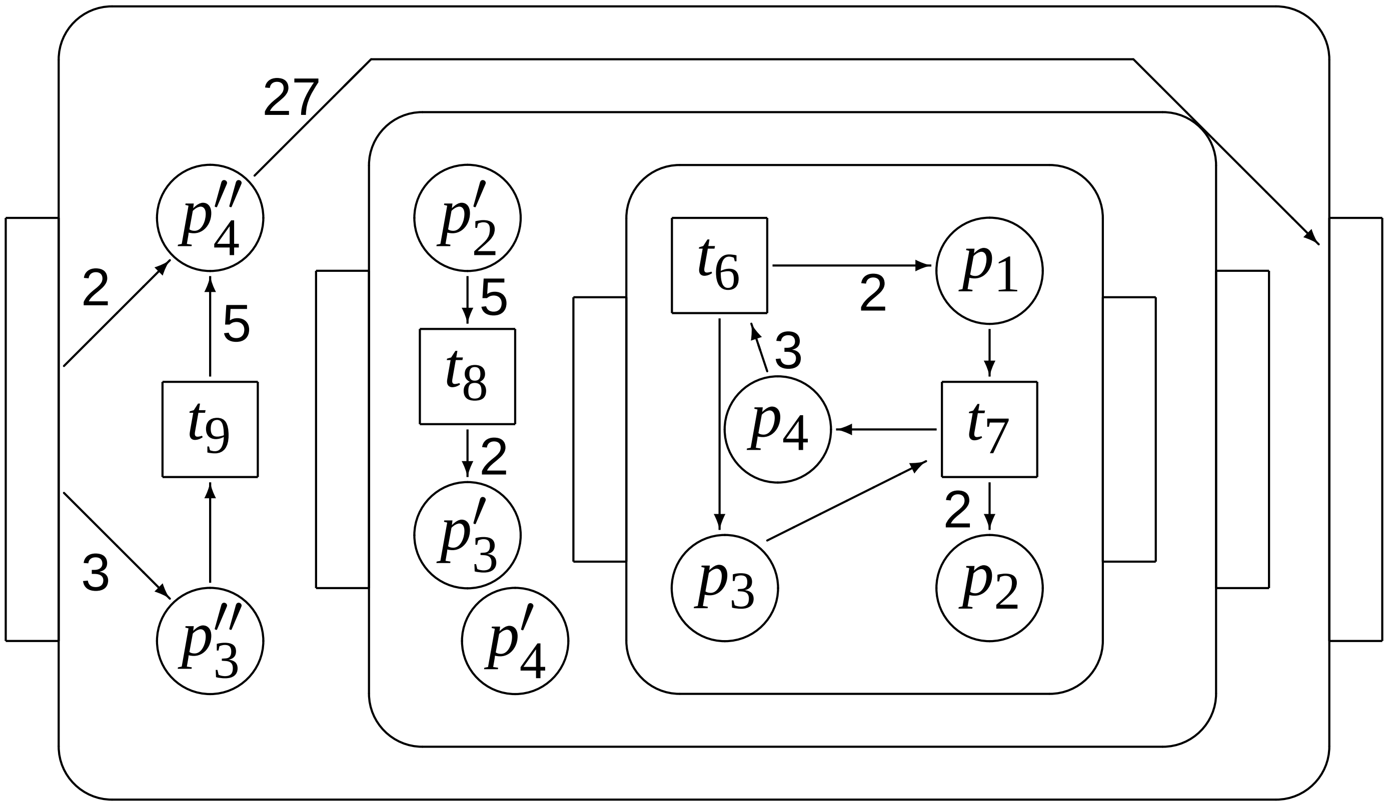
$$\exists \mathbf{g} \in \mathbb{N}_+^P \; \forall p,p' \in P \; \mathbf{g}(p) \leq \mathbf{g}(p') \rightarrow (\forall t \in T \; (p',t) \in I \rightarrow (p,t) \in I),$$

is decidable.

Example: Start marking: $\{p_3 \mapsto 3, p_4 \mapsto 2\}$, end marking $\{p_4 \mapsto 27\}$



We find $\mathbf{g}$ with $\mathbf{g}(p_1) = 1$, $\mathbf{g}(p_2) = 2$ and $\mathbf{g}(p_3) = \mathbf{g}(p_4) = 3$ and construct $T_1 = \{t_6, t_7\}$ with $\mathbf{R}(T_1) = \left\{ \left[\frac{p_4^-}{3}, \frac{p_1^+}{2}, \frac{p_3^+}{1} \right], \left[\frac{p_1^-}{1}, \frac{p_3^-}{1}, \frac{p_2^+}{2}, \frac{p_4^+}{1} \right] \right\} + \mathbf{I}d_P$, as innermost net of



This enables the firing sequence $w = t_6t_7t_7$ from $\left[\begin{smallmatrix}p_3\\1\end{smallmatrix}, \begin{smallmatrix}p_4\\3\end{smallmatrix}\right]$ to $\left[\begin{smallmatrix}p_2\\4\end{smallmatrix}, \begin{smallmatrix}p_4\\2\end{smallmatrix}\right]$ on the innermost level as $\left[\begin{smallmatrix}p_3^-\\1\end{smallmatrix}, \begin{smallmatrix}p_4^-\\3\end{smallmatrix}, \begin{smallmatrix}p_2^+\\4\end{smallmatrix}, \begin{smallmatrix}p_4^+\\2\end{smallmatrix}\right] \in \mathbf{R}(*_{P_{T_1}}(T_1)) = \mathbf{R}(t_2) \subseteq \mathbf{R}(T_2)$. Together with

$\begin{bmatrix} p_2^- & p_3^+ \\ 5 & 2 \end{bmatrix} \in \mathbf{R}(T_2)$ for t_8, we get the firing sequence $w' = (w)(w)t_8(w)t_8(w)t_8(w)t_8$ from $\begin{bmatrix} p_3, & p_4 \\ 2 & 7 \end{bmatrix}$ to $\begin{bmatrix} p_3, & p_4 \\ 5 & 2 \end{bmatrix}$ on the next level as $\begin{bmatrix} p_3^- & p_4^- & p_3^+ & p_4^+ \\ 2 & 7 & 5 & 2 \end{bmatrix} \in \mathbf{R}(*_{P_{T_2}}(T_2)) = \mathbf{R}(t_3) \subseteq \mathbf{R}(T_3)$. Together with $\begin{bmatrix} p_3^- & p_4^+ \\ 1 & 5 \end{bmatrix} \in \mathbf{R}(T_3)$ for t_9, this enables the firing sequence $w'' = t_9(w')t_9^5$ from $\begin{bmatrix} p_3, & p_4 \\ 3 & 2 \end{bmatrix}$ to $\begin{bmatrix} p_4 \\ 27 \end{bmatrix}$ on the following level as $\begin{bmatrix} p_3^- & p_4^- & p_4^+ \\ 3 & 2 & 27 \end{bmatrix} \in \mathbf{R}(*_{P_{T_3}}(T_3)) = \mathbf{R}(t_4) = \mathbf{R}(T_4)$.

Priority-Multicounter-Automata

A *priority-multicounter-automaton* is a one-way automaton described by the 6-tuple

$$A = (k, Z, \Sigma, \delta, z_0, E)$$

with the set of states Z, the input alphabet Σ, the transition relation

$$\delta \subseteq (Z \times (\Sigma \cup \{\lambda\}) \times \{0 \ldots k\}) \times (Z \times \{-1, 0, 1\}^k),$$

initial state z_0, the accepting states $E \subseteq Z$, the set of configurations $C_A = Z \times \Sigma^* \times \mathbb{N}^k$, the initial configuration $\sigma_A(x) = \langle z_0, x, \underbrace{0, \ldots, 0}_{k} \rangle$ and configuration transition relation

$$\langle z, ax, n_1, \ldots, n_k \rangle \;\underset{A}{\vdash}\; \langle z', x, n_1 + i_1, \ldots, n_k + i_k \rangle$$

if and only if $z, z' \in Z, a \in \Sigma \cup \{\lambda\}, \langle (z, a, j), (z', i_1, \ldots i_k) \rangle \in \delta$, $\forall i \leq j \; n_i = 0$.

Recognized language:

$L(A) = \{w \mid \exists z_e \in E \; \exists n_1, \ldots, n_k \in \mathbb{N} \; \langle z_0, w, 0, \ldots, 0 \rangle \; \vdash_A^* \; \langle z_e, \lambda, n_1, \ldots, n_k \rangle$.

Alternatively $L(A) = \{w \mid \langle z_0, w, 0, \ldots, 0 \rangle \; \vdash_A^* \; \langle z_e, \lambda, 0, \ldots, 0 \rangle\}$.

Using Theorem 2, we get:

Theorem 3 *The emptiness problem for priority-multicounter-automata is decidable.*

The same holds for the halting problem by constructing an automaton which contains its input in the states.

Restricted Priority- Multipushdown- Automata

Different treatment of one of the two pushdown symbols $\{0, 1\}$:

A 0 can be pushed to and popped from every pushdown store independently, but a 1 can only be pushed to or popped from a pushdown store if all pushdown stores with a lower order are empty.

Restriction: If a 1 is popped from a pushdown store, then a 1 cannot be pushed anymore to this store until it is empty.

Theorem 4 *The emptiness problem for restricted priority-multipushdown-automata is decidable.*

This generalizes the result in [JKLP90] that $\underline{LIN}\%{D'_1}^*$ (the class of languages generated by linear grammar and deletion of semi Dyck words) is recursive.

Conjecture: Decidability still holds in the unrestricted case.

Open problem: Pushdown automaton with additional weak counters (without zero-test).