# IDEAL TOPOLOGIES, LOCAL COHOMOLOGY AND CONNECTEDNESS

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ABSTRACT. Let $\mathfrak{a}$ be an ideal of a local ring $(R, \mathfrak{m})$ and let $N$ be a finitely generated $R$-module of dimension $d$. It is shown that $H^d_{\mathfrak{a}}(N) \simeq H^d_{\mathfrak{m}}(N)/\sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^n)$, where for an Artinian $R$-module $X$ we put $\langle\mathfrak{m}\rangle X = \cap_{n\in\mathbb{N}}\mathfrak{m}^n X$. As a consequence several vanishing and connectedness results are deduced.

## 1. Introduction

The following article is concerned with the local cohomology modules $H^i_{\mathfrak{a}}(N), i \in \mathbb{Z}$, of a finitely generated $R$-module $N$, where $\mathfrak{a}$ denotes an ideal of a local Noetherian ring $(R, \mathfrak{m})$. Particularly we are interested in the case where $i = \dim_R N$, the last possible non-vanishing local cohomology module. This is known by the vanishing result of Grothendieck, saying that $H^i_{\mathfrak{a}}(N) = 0$ for all $i > \dim_R N$; see [5, 1.12] for the details. In the following we refer to [5] and the textbook [2] for the definitions and basic results about local cohomology.

To be more precise, for a finitely generated $R$-module $N$ there is the long exact sequence

$$\ldots \to H^i_{\mathfrak{m}}(N) \to H^i_{\mathfrak{a}}(N) \to \varinjlim \operatorname{Ext}^i_R(\mathfrak{m}^n/\mathfrak{a}^n, N) \to \ldots$$

relating the local cohomology of $N$ with respect to $\mathfrak{a}$ and $\mathfrak{m}$ resp. It follows by Hartshorne's result, see [7, p. 417], that $\varinjlim \operatorname{Ext}^d_R(\mathfrak{m}^n/\mathfrak{a}^n, N) = 0$ in the case $d = \dim_R N$. Therefore $H^d_{\mathfrak{a}}(N)$ is – as an epimorphic image of $H^d_{\mathfrak{m}}(N)$ – an Artinian $R$-module. See also [17] for a different argument and results about its attached prime ideals. In fact, the vanishing of $H^d_{\mathfrak{a}}(N)$ – known as the Lichtenbaum-Hartshorne vanishing theorem – plays a central rôle in different applications, among them those related to connectivity as we will derive in the fourth section.

As the main result of the third section we calculate the kernel of the natural epimorphism $H^d_{\mathfrak{m}}(N) \to H^d_{\mathfrak{a}}(N)$. More precisely we prove the following result:

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**Theorem 1.1.** *Let $\mathfrak{a}$ denote an ideal of a local ring $(R, \mathfrak{m})$. Let $N$ be a finitely generated $R$-module with $d = \dim_R N$. Then there is a functorial isomorphism*

$$H^d_{\mathfrak{a}}(N) \simeq H^d_{\mathfrak{m}}(N) / \sum_{n \in \mathbb{N}} \langle \mathfrak{m} \rangle (0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^n).$$

*(For an Artinian $R$-module $X$ we put $\langle \mathfrak{m} \rangle X = \cap_{n \in \mathbb{N}} \mathfrak{m}^n X$.)*

The result in 1.1 is proved in 3.2. For an Artinian $R$-module $A$ there is the theory of secondary representations; see [10] for the details. In particular, for an ideal $\mathfrak{a}$ of $R$ it follows that $\langle \mathfrak{a} \rangle A = \mathfrak{a}^m A, m \gg 0$, coincides with the sum of all $\mathfrak{p}_i$-secondary components $A_i$ of a minimal secondary representation $A = \sum_{i=1}^n A_i$ of $A$ such that $\mathfrak{a} \not\subseteq \mathfrak{p}_i$ (where $\mathfrak{p}_i = \operatorname{Rad}(0 :_R A_i), 1 \leq i \leq n$). Pursuing this point of view further we derive the following consequence of 1.1.

**Corollary 1.2.** *With the previous notation and those of 1.1 let $H^d_{\mathfrak{m}}(N) = \sum_{i=1}^n A_i$ denote a minimal secondary representation of $H^d_{\mathfrak{m}}(N)$ as an $\hat{R}$-module. Suppose that* $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p}_i = 0$ *if and only if* $1 \leq i \leq m$ *for a certain integer* $0 \leq m \leq n$. *Then*

$$H^d_{\mathfrak{a}}(N) \simeq \sum_{i=1}^m (A_i + B)/B, \quad \textit{where} \;\; B = \sum_{i=m+1}^n A_i,$$

*is a minimal secondary representation of $H^d_{\mathfrak{a}}(N)$. In particular $H^d_{\mathfrak{a}}(N) = 0$ if and only if* $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} > 0$ *for all* $\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{N}$ *with* $\dim \hat{R}/\mathfrak{p} = d$.

Therefore we obtain not only a description of the attached prime ideals, as done in [17], but we have also a minimal secondary representation of $H^d_{\mathfrak{a}}(N)$. Corollary 1.2 is proved in Section 3: see 3.3.

A main technique in most of the known proofs of the Lichtenbaum-Hartshorne vanishing theorems is the equivalence of a certain topology to the adic topology: see e.g. the second author's approach in [15]. Here we have to introduce a new subject concerning the topology of Artinian modules. To this end we have to establish a certain dual to a Theorem of Chevalley for Artinian modules; see 2.1. As an application we have the following form of the Lichtenbaum-Hartshorne vanishing theorem. To this end put $K(N) = \operatorname{Hom}_R(H^d_{\mathfrak{m}}(N), E), d = \dim_R N$, where $E$ denotes the injective hull of the residue field of $R$.

**Theorem 1.3.** *Let $\mathfrak{a}$ denote an ideal of a local ring $(R, \mathfrak{m})$. For a finitely generated $R$-module $N$ with $d = \dim_R N$ the following conditions are equivalent:*

(i) $H^d_{\mathfrak{a}}(N) = 0$.

(ii) $\sum_{n \in \mathbb{N}} \langle \mathfrak{m} \rangle (0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^n) = H^d_{\mathfrak{m}}(N)$.

(iii) *For any integer $m \in \mathbb{N}$ there exists an $n = n(m) \in \mathbb{N}$ such that*

$$0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^m \subseteq \langle \mathfrak{m} \rangle (0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^n).$$

(iv) $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} > 0$ *for all* $\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{N}$ *such that* $\dim \hat{R}/\mathfrak{p} = d$.

(v) *For any intger* $m \in \mathbb{N}$ *there exists an* $n = n(m) \in \mathbb{N}$ *such that*

$$\mathfrak{a}^n K(N) :_{K(N)} \langle \mathfrak{m} \rangle \subseteq \mathfrak{a}^m K(N).$$

The proof of Theorem 1.3 is given in 3.4. It was discovered by Brodmann and Rung, see [1], that the Lichtenbaum-Hartshorne vanishing theorem yields connectedness theorems. In Section 4 we apply our refinements about the Lichtenbaum-Hartshorne vanishing theorem in order to generalize those connectedness results. Mainly we are able to generalize the known results for local rings to finitely generated modules.

Our terminology follows that of the textbook [2].

## 2. Ideal Topologies on Artinian Modules

In this section let $A$ denote an Artinian $R$-module, where $R$ denotes a commutative ring. For an ideal $\mathfrak{a}$ of the ring $R$ consider the increasing sequence of submodules $\{0 :_A \mathfrak{a}^n\}_{n\in\mathbb{N}}$ of $A$. In the sequel let us relate this sequence to an arbitrary increasing sequence $\{A_n\}_{n\in\mathbb{N}}$ of submodules of $A$. To this end we prove a certain variation of Chevalley's Theorem for an Artinian $R$-module $A$.

**Lemma 2.1.** *Let* $\{A_n\}_{n\in\mathbb{N}}$ *denote an increasing sequence of submodules of an Artinian* $R$*-module* $A$*. Let* $\mathfrak{a}$ *be an ideal of* $R$*. Suppose that the following conditions are satisfied:*

a) $\sum_{n\in\mathbb{N}} A_n = A$, *and*

b) *for any fixed integer* $m \in \mathbb{N}$ *the sequence* $\{(0 :_A \mathfrak{a}^m) \cap A_n\}_{n\in\mathbb{N}}$ *satisfies the ascending chain condition.*

*Then, for any fixed* $m \in \mathbb{N}$ *there exists an integer* $n = n(m)$ *such that* $0 :_A \mathfrak{a}^m \subseteq A_n$.

*Proof.* Let $m \in \mathbb{N}$ denote a given integer. In view of b) there exists an integer $n = n(m)$ such that

$$(0 :_A \mathfrak{a}^m) \cap A_n = (0 :_A \mathfrak{a}^m) \cap A_{n+l}$$

for all $l \geq 1$. Denote $(0 :_A \mathfrak{a}^m) \cap A_n$ by $B_m$. It follows from a) that

$$0 :_A \mathfrak{a}^m = \sum_{n\in\mathbb{N}} (0 :_A \mathfrak{a}^m) \cap A_n.$$

Because of the definition of $B_m$ this means $0 :_A \mathfrak{a}^m = B_m$ and $0 :_A \mathfrak{a}^m \subseteq A_n$ for a certain $n = n(m)$, as required. □

In the sequel we need a few auxiliary results on Artinian modules. For the sake of completeness we summarize them in the following remark.

**Remark 2.2.** a) Let $B$ denote a submodule of an Artinian $R$-module $A$. For an ideal $\mathfrak{a}$ of $R$ there exists an integer $c$ such that

$$(0 :_A \mathfrak{a}^{n+c}) + B = ((0 :_A \mathfrak{a}^c) + B) :_A \mathfrak{a}^n$$

for all $n \geq 0$; see [9, Proposition 3]. This is the dual of the Artin-Rees Lemma for Noetherian modules.
b) It is well-known that for a non-zero Artinian $R$-module $A$ its support $\operatorname{Supp}_R A$ consists of finitely many maximal ideals, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Put $\mathfrak{M} = \mathfrak{m}_1 \cap \ldots \cap \mathfrak{m}_n$ and $S = \varprojlim R/\mathfrak{M}^i$. By the work of Sharp, see e.g. [18], it follows that $A$ has a natural structure as a module over $R' := S/\operatorname{Ann}_S A$. Moreover $R'$ is a complete semi-local Noetherian ring. Let $\psi : R \to R'$ denote the natural homomorphism. For any element $r \in R$ the multiplication by $r$ on $A$ has the same effect as the multiplication by $\psi(r) \in R'$ on $A$ as an $R'$-module. Furthermore a subset of $A$ is an $R$-module if and only if it is an $R'$-submodule.

In the following we need a few basic results on the module structure of an Artinian $R$-module over $R'$ and $S$ respectively. To this end put $E = \oplus_{i=1}^n E(R'/\mathfrak{m}'_i)$. It is the minimal injective co-generator of $R'$. Here $E(R'/\mathfrak{m}'_i)$ denotes the injective hull of $R'/\mathfrak{m}'_i, i = 1, \ldots, n$. Then $D(\cdot) = \operatorname{Hom}_{R'}(\cdot, E)$ denotes the Matlis duality functor on the complete semi-local Noetherian ring $R'$.

**Lemma 2.3.** *With the previous notation let $B$ denote a subquotient of $A$. Then*

$$\operatorname{Rad}(\operatorname{Ann}_S(0 :_B \mathfrak{a})) = \operatorname{Rad}(\operatorname{Ann}_S B + \mathfrak{a}S)$$

*for any ideal $\mathfrak{a}$ of $R$.*

*Proof.* By Matlis duality (see [13, Theorem 1.6] for the case of a complete semi-local ring) it follows that $D(B)$ is a Noetherian module. Therefore

$$\operatorname{Rad}(\operatorname{Ann}_{R'}(D(B) \otimes_{R'} R'/\mathfrak{a}R')) = \operatorname{Rad}(\operatorname{Ann}_{R'} D(B) + \mathfrak{a}R'),$$

as is well-known for a Noetherian $R'$-module $D(B)$. Now we have that $\operatorname{Ann}_{R'} D(X) = \operatorname{Ann}_{R'} X$ for an arbitrary $R'$-module $X$. Because of $D(0 :_B \mathfrak{a}) \simeq D(B) \otimes_{R'} R'/\mathfrak{a}R'$ the claim follows now. $\square$

For an ideal $\mathfrak{a}$ of $R$ and an Artinian $R$-module $A$ the descending sequence of submodules

$$A \supseteq \mathfrak{a}A \supseteq \ldots \supseteq \mathfrak{a}^n A \supseteq \ldots$$

becomes stationary. Denote its ultimate constant value by $\langle\mathfrak{a}\rangle A$, so that $\langle\mathfrak{a}\rangle A = \mathfrak{a}^n A$ for all large $n \in \mathbb{N}$. It follows, see [10], that $\langle\mathfrak{a}\rangle A$ coincides with the sum of all those $\mathfrak{p}$-secondary components of the minimal secondary representation of $A$ such that $\mathfrak{a} \not\subseteq \mathfrak{p}$.

Now we are prepared in order to prove the main theorem of this section. In fact there is a characterization of the equivalence of certain increasing families of submodules of an Artinian module.

**Theorem 2.4.** *Let $\mathfrak{M}$ and $S$ be as in 2.2 b). Let $\mathfrak{a} \subseteq \mathfrak{b}$ denote two ideals of $R$. For an Artinian $R$-module $A$ the following conditions are equivalent:*

(i) *For any $m \in \mathbb{N}$ there is an integer $n = n(m)$ such that $0 :_A \mathfrak{M}^m \subseteq \langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n)$.*
(ii) $\sum_{n\in\mathbb{N}} \langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n) = A$.
(iii) $\operatorname{Rad}(\mathfrak{p} + \mathfrak{a}S) \subsetneq \operatorname{Rad}(\mathfrak{p} + \mathfrak{b}S)$ *for all* $\mathfrak{p} \in \operatorname{Att}_S A$.

*Proof.* Because $A$ is an Artinian $R$-module any element of $A$ is annihilated by a certain power of $\mathfrak{M}$. Therefore it follows that $A = \sum_{n\in\mathbb{N}} 0 :_A \mathfrak{M}^n$. Now we put

$$A_n = \langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n) \text{ for any } n \in \mathbb{N}.$$

Because $0 :_A \mathfrak{M}^n$ is a Noetherian submodule of $A$ the equivalence of the conditions (i) and (ii) is a consequence of 2.1.

In order to prove the implication (ii) $\Rightarrow$ (iii) assume the existence of an attached prime ideal $\mathfrak{p} \in \operatorname{Att}_S A$ such that $\operatorname{Rad}(\mathfrak{p} + \mathfrak{a}S) = \operatorname{Rad}(\mathfrak{p} + \mathfrak{b}S)$. Because $\mathfrak{p}$ is an attached prime ideal of $A$ it follows that $\langle \mathfrak{p} \rangle A \neq A$. Moreover

$$\langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n) \subseteq \langle \mathfrak{p} + \mathfrak{b}S \rangle (0 :_A \mathfrak{a}^n) = \langle \mathfrak{p} + \mathfrak{a}S \rangle (0 :_A \mathfrak{a}^n) = \langle \mathfrak{p} \rangle (0 :_A \mathfrak{a}^n)$$

for all $n \in \mathbb{N}$. Therefore the containment relation

$$A = \sum_{n\in\mathbb{N}} \langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n) \subseteq \sum_{n\in\mathbb{N}} \langle \mathfrak{p} \rangle (0 :_A \mathfrak{a}^n) \subseteq \langle \mathfrak{p} \rangle A$$

provides a contradiction.

Finally we have to prove the implication (iii) $\Rightarrow$ (ii). Suppose the contrary is true. Then there is an attached prime ideal

$$\mathfrak{p} \in \operatorname{Att}_S(A / \sum_{n\in\mathbb{N}} \langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n)).$$

So there exists a submodule $B$ of $A$ containing $\sum_{n\in\mathbb{N}} \langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n)$ such that $\mathfrak{p} = \operatorname{Ann}_S(A/B)$. By the dual of the Artin-Rees Lemma for Artinian modules, see 2.2 a), there is an integer $l \in \mathbb{N}$ such that

$$B :_A \mathfrak{a} \subseteq (0 :_A \mathfrak{a}^l) + B.$$

Because of the choice of $B$ it follows that $\langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^l) \subseteq B$. Choose $m \in \mathbb{N}$ an integer such that $\langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^l) = \mathfrak{b}^n (0 :_A \mathfrak{a}^l)$ for all $n \geq m$. But then it follows that $(0 :_A \mathfrak{a}^l) + B \subseteq B :_A \mathfrak{b}^m$ and therefore $B :_A \mathfrak{a} \subseteq B :_A \mathfrak{b}^m$. This implies

$$\operatorname{Ann}_S(0 :_{A/B} \mathfrak{b}^m) \subseteq \operatorname{Ann}_S(0 :_{A/B} \mathfrak{a}).$$

Because of $\operatorname{Ann}_S(A/B) = \mathfrak{p}$ Lemma 2.3 provides a contradiction. □

In the particular case where $(R, \mathfrak{m})$ is a quasi-local ring and $\mathfrak{b} = \mathfrak{m}$ the previous result has the following application.

**Corollary 2.5.** *Let $\mathfrak{a}$ denote an ideal of a quasi-local ring $(R, \mathfrak{m})$. For an Artinian $R$-module $A$ the following conditions are equivalent:*

(i) *For any $m \in \mathbb{N}$ there is an integer $n = n(m)$ such that $0 :_A \mathfrak{a}^m \subseteq \langle \mathfrak{m} \rangle (0 :_A \mathfrak{a}^n)$.*
(ii) $\sum_{n \in \mathbb{N}} \langle \mathfrak{m} \rangle (0 :_A \mathfrak{a}^n) = A$.
(iii) $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} > 0$ *for all* $\mathfrak{p} \in \operatorname{Att}_{\hat{R}} A$.

*Proof.* Since $A$ is an Artinian $R$-module it follows that $\sum_{n \in \mathbb{N}} (0 :_A \mathfrak{m}^n) = A$. Because $(0 :_A \mathfrak{m}^n) \subseteq (0 :_A \mathfrak{a}^n)$ for all $n \in \mathbb{N}$ the condition (i) implies (ii).

For a fixed integer $m \in \mathbb{N}$ the factor module $(0 :_A \mathfrak{a}^m)/\langle \mathfrak{m} \rangle (0 :_A \mathfrak{a}^m)$ is a Noetherian $R$-module. Put $A_n = \langle \mathfrak{m} \rangle (0 :_A \mathfrak{a}^n)$. Then the sequence of submodules $\{(0 :_A \mathfrak{a}^m) \cap A_n\}_{n \in \mathbb{N}}$ satisfies the ascending chain condition. By 2.1 it follows that condition (ii) implies (i).

Let $(R', \mathfrak{m}')$ denote the quasi-local ring introduced in 2.2 b). By virtue of [18, 1.2] it turns out that $\mathfrak{m}R'$ is an $\mathfrak{m}'$-primary ideal. Therefore $\operatorname{Rad}(\mathfrak{p} + \mathfrak{m}\hat{R}) = \hat{\mathfrak{m}}$ for any attached prime ideal $\mathfrak{p} \in \operatorname{Att}_{\hat{R}} A$. Recall that $\operatorname{Ann}_{\hat{R}} A$ is contained in every element of $\operatorname{Att}_{\hat{R}} A$. By 2.4 this establishes the equivalence between the conditions (ii) and (iii). □

The following example shows that it is impossible to replace condition (iii) of 2.5 by the corresponding statement on the ring itself.

**Example 2.6.** Let $(R, \mathfrak{m})$ denote a two-dimensional local domain such that the completion $\hat{R}$ possesses a one-dimensional embedded prime ideal $\mathfrak{q}$ : see the Example 2 in [12, Appendix] for a concrete realization. Let $\mathfrak{a}$ denote a one-dimensional ideal of $R$. For $E = E(R/\mathfrak{m})$, the injective hull of the residue field it follows that

$$\operatorname{Att}_{\hat{R}} E = \operatorname{Ass} \hat{R} \quad \text{and} \quad \operatorname{Att}_R E = \{0\}.$$

Therefore $\dim R/\mathfrak{a} + \mathfrak{p} > 0$ for all $\mathfrak{p} \in \operatorname{Att}_R E$. Furthermore we have that $\mathfrak{a}\hat{R} \not\subseteq \mathfrak{q}$ because otherwise we would have

$$\mathfrak{a} = \mathfrak{a}\hat{R} \cap R \subseteq \mathfrak{q} \cap R = (0),$$

a contradiction. Hence $\mathfrak{a}\hat{R} + \mathfrak{q}$ is an $\mathfrak{m}\hat{R}$-primary ideal. But on the other hand we get

$$\sum_{n \in \mathbb{N}} \langle \mathfrak{m} \rangle (0 :_E \mathfrak{a}^n) = \sum_{n \in \mathbb{N}} \langle \mathfrak{q} \rangle (0 :_E \mathfrak{a}^n \hat{R}) \subseteq \langle \mathfrak{q} \rangle E$$

and $\langle \mathfrak{q} \rangle E \neq E$ because $\mathfrak{q} \in \operatorname{Att}_{\hat{R}} E$.

Let $N$ denote a Noetherian $R$-module. Let $\mathfrak{a}$ be an ideal of $R$. For a submodule $M$ of $N$ consider the increasing sequence of submodules

$$M \subseteq M :_N \mathfrak{a} \subseteq \ldots \subseteq M :_N \mathfrak{a}^n \subseteq \ldots .$$

Its ultimate constant value is denoted by $M :_N \langle \mathfrak{a} \rangle$. Therefore $M :_N \langle \mathfrak{a} \rangle = M :_N \mathfrak{a}^n$ for all large $n \in \mathbb{N}$.

**Remark 2.7.** In [16, Corollary 2.2] the following result was shown for the case where $N = R$ :

Let $\mathfrak{a}$ denote an ideal of a local Noetherian ring $(R, \mathfrak{m})$. For a finitely generated $R$-module $N$ the following conditions are equivalent:

(i) For an integer $m \in \mathbb{N}$ there is an integer $n = n(m)$ such that $\mathfrak{a}^n N :_N \langle \mathfrak{m} \rangle \subseteq \mathfrak{a}^m N$.
(ii) $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} > 0$ for all $\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{N}$.

It is easy to extend the original proof 'mutatis mutantis' to this more general case. On the other hand it is easily seen that this equivalence is a consequence of our result in 2.5. To this end we have to apply the faithfully flat extension $\hat{R}$ and the Matlis duality for a complete local ring. So our approach to the study of certain filtrations on Artinian modules may be viewed as the dual to the study of ideal topologies of finitely generated modules in [16].

In 2.4 we have characterized the situation whenever the submodule $\sum_{n \in \mathbb{N}} \langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n)$ of an Artinian $R$-module $A$ coincides with $A$. In the following we will generalize this result to an explicit description of it. To this end let $A = \sum_{i=1}^n A_i$ denote a minimal secondary representation of $A$ as an $S$-module. In particular $A_i$ is a $\mathfrak{p}_i$-secondary submodule of $A$.

**Theorem 2.8.** *Let $\mathfrak{a} \subseteq \mathfrak{b}$ denote two ideals of $R$. Let $S$ denote the ring introduced in 2.2 b). Define*

$$V = \{\mathfrak{p}_i \in \operatorname{Att}_S A \mid \operatorname{Rad}(\mathfrak{p}_i + \mathfrak{a}S) \subsetneq \operatorname{Rad}(\mathfrak{p}_i + \mathfrak{b}S)\}.$$

*Then $\sum_{\mathfrak{p}_i \in V} A_i$ is a minimal secondary representation of $\sum_{n \in \mathbb{N}} \langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n)$.*

*Proof.* As before put $\operatorname{Att}_S A = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$. Without loss of generality we may assume that $V = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}$ for a certain integer $0 \leq m \leq n$. Put $B = \sum_{i=1}^m A_i$ and $C = \sum_{i=m+1}^n A_i$. By 2.4 applied to the submodule $B$ it follows that $B = \sum_{n \in \mathbb{N}} \langle \mathfrak{b} \rangle (0 :_B \mathfrak{a}^n)$. Hence

$$B = \sum_{n \in \mathbb{N}} \langle \mathfrak{b} \rangle (0 :_B \mathfrak{a}^n) \subseteq \sum_{n \in \mathbb{N}} \langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n).$$

Therefore in order to prove the claim it will be enough to show that

$$\langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^l) \subseteq B \text{ for all } l \in \mathbb{N}.$$

Now we put $\mathfrak{c} = \cap_{i=m+1}^{n} \mathfrak{p}_i$. Then it turns out that $\operatorname{Rad}(\mathfrak{c}+\mathfrak{b}S) = \operatorname{Rad}(\mathfrak{c}+\mathfrak{a}S)$. Since $S/\operatorname{Ann}_S A$ is a Noetherian ring there exists an integer $l \in \mathbb{N}$ such that $\mathfrak{c}^l C = 0$. Therefore we derive the following containment relations

$$\langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n) \subseteq \langle \mathfrak{b}S + \mathfrak{c} \rangle (0 :_A \mathfrak{a}^n) = \langle \mathfrak{a}S + \mathfrak{c} \rangle (0 :_A \mathfrak{a}^n) \subseteq \mathfrak{c}^l B.$$

So we get that $\langle \mathfrak{b} \rangle (0 :_A \mathfrak{a}^n) \subseteq B$, which completes the proof. □

Let $U \subset R$ denote a multiplicatively closed subset of $R$. The module $\operatorname{Hom}_R(R_U, A)$ is called the co-localization of $A$ with respect to $U$; see [11] for some more details. There is a natural homomorphism

$$\operatorname{Hom}_R(R_U, A) \to A, \ f \mapsto f(1).$$

Let $A$ denote an Artinian $R$-module with $A = \sum_{i=1}^{n} A_i$ a minimal secondary representation, where $A_i$ denotes the $\mathfrak{p}_i$-secondary component. By [11, Corollary 3.3] it follows that the image of the natural homomorphism $\operatorname{Hom}_R(R_U, A) \to A$ coincides with the $U$-component of $A$, i.e. $U(A) = \cap_{u \in U} uA = \sum_{\mathfrak{p}_i \cap U = \emptyset} A_i$.

**Remark 2.9.** Let $(R, \mathfrak{m})$ denote a complete quasi-local ring. Then there will be another expression for

$$\sum_{n \in \mathbb{N}} \langle \mathfrak{m} \rangle (0 :_A \mathfrak{a}^n)$$

for an Artinian $R$-module $A$ and an ideal $\mathfrak{a}$ of $R$. To this end put

$$W = \{\mathfrak{p} \in \operatorname{Att}_R A \mid \dim R/\mathfrak{a}R + \mathfrak{p} > 0\}.$$

Then we have that

$$\sum_{n \in \mathbb{N}} \langle \mathfrak{m} \rangle (0 :_A \mathfrak{a}^n) = \operatorname{Im}(\operatorname{Hom}_R(R_U, A) \to A),$$

where $U = \cap_{\mathfrak{p} \in W} (R \setminus \mathfrak{p})$. Therefore the result in 2.5 describes the surjectivity of the natural map $\operatorname{Hom}_R(R_U, A) \to A$.

## 3. On the Lichtenbaum-Hartshorne Theorem

In this section let $\mathfrak{a}$ denote an ideal of a local Noetherian ring $(R, \mathfrak{m})$. In the following we investigate the local cohomology modules $H^i_{\mathfrak{m}}(N)$ resp. $H^i_{\mathfrak{a}}(N)$ of a finitely generated $R$-module $N$ of dimension $d := \dim_R N$. Mainly we are interested in the case where $i = d$. In fact for $i = d$ we will describe a close relation between them. Before we investigate the local cohomology modules we need another preliminary result about Artinian $R$-modules.

**Lemma 3.1.** *Let $\mathfrak{a}$ denote an ideal of a local ring $(R, \mathfrak{m})$. Let $A$ be an Artinian $R$-module. Then there is a functorial isomorphism*

$$(A/\sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_A \mathfrak{a}^n)) \otimes_R N \simeq (A \otimes_R N)/\sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_{A\otimes_R N} \mathfrak{a}^n)$$

*for any finitely generated $R$-module $N$.*

*Proof.* Let $U$ denote a multiplicatively closed subset of $R$. Since $A$ is an Artinian $R$-module, this is also true for $A \otimes_R N$. So there exists an element $u \in U$ such that $U(A) = uA$ and $U(A \otimes_R N) = u(A \otimes_R N)$. Therefore there are the functorial isomorphisms

$$A/U(A)\otimes_R N \simeq (R/uR\otimes_R A)\otimes_R N \simeq (A\otimes_R N)/u(A\otimes_R N) = (A\otimes_R N)/U(A\otimes_R N).$$

Let $A = \sum_{i=1}^n A_i$ denote a minimal secondary representation of $A$ considered as an $\hat{R}$-module. We may assume that $\operatorname{Att}_{\hat{R}} A = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ is ordered in such a way that $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p}_i > 0$ if and only if $1 \leq i \leq m$ for a certain integer $0 \leq m \leq n$. Now we put $U = \hat{R} \setminus \cup_{i=1}^m \mathfrak{p}_i$. By virtue of 2.8 it follows that

$$U(A) = \sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_A \mathfrak{a}^n).$$

Let $\phi_i : A_i \otimes_R N \to A \otimes_R N$ be the natural homomorphism induced by the inclusion map $A_i \hookrightarrow A$. Then $\operatorname{Im} \phi_i$ is either zero or a $\mathfrak{p}_i$-secondary submodule of $A \otimes_R N$. Thus

$$A \otimes_R N = \sum_{i=1}^n \operatorname{Im} \phi_i$$

is a secondary representation of $A \otimes_R N$ and $\operatorname{Att}_{\hat{R}}(A \otimes_R N) \subseteq \operatorname{Att}_{\hat{R}} A$ because any secondary representation can be reduced to a minimal one. Again in view of 2.8 it follows that

$$U(A \otimes_R N) = \sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_{A\otimes_R N} \mathfrak{a}^n).$$

Since any Artinian $R$-module is an Artinian module over $\hat{R}$ in a natural way, one may easily reduce the situation to the case that $R$ is complete. Hence from the previous isomorphism we get the isomorphism of the statement. □

We refer to Grothendieck's Lecture Note [5] resp. to the textbook [2] for the basic results on local cohomology. In particular, $H^i_{\mathfrak{m}}(N), i \in \mathbb{Z}$, is an Artinian $R$-module for a finitely generated $R$-module $N$. Moreover let $E = E(R/\mathfrak{m})$ denote the injective hull of the residue field $k = R/\mathfrak{m}$ of $R$.

**Theorem 3.2.** *Let $\mathfrak{a}$ denote an ideal of $(R, \mathfrak{m})$. For a finitely generated $R$-module $N$ there is a functorial isomorphism*

$$H^d_{\mathfrak{a}}(N) \simeq H^d_{\mathfrak{m}}(N)/\sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^n),$$

*where $d = \dim_R N$. In particular $H^d_{\mathfrak{a}}(N)$ is an Artinian $R$-module.*

*Proof.* In the first step we show the claim in the case of $N = R$, a local Gorenstein ring of dimension $d = \dim R$. By the expression of local cohomology as the direct limit of Ext-modules and the Local Duality Theorem, see e.g. [5], there are the natural isomorphisms

$$H^d_{\mathfrak{a}}(R) \simeq \varinjlim \operatorname{Ext}^d_R(R/\mathfrak{a}^n, R) \simeq \varinjlim \operatorname{Hom}_R(H^0_{\mathfrak{m}}(R/\mathfrak{a}^n), E).$$

Because $H^0_{\mathfrak{m}}(R/\mathfrak{a}^n) \simeq \operatorname{Hom}_R(R/\mathfrak{m}^t, R/\mathfrak{a}^n)$ for all large $t \in \mathbb{N}$ there is an isomorphism

$$\operatorname{Hom}_R(H^0_{\mathfrak{m}}(R/\mathfrak{a}^n), E) \simeq (0 :_E \mathfrak{a}^n)/\langle\mathfrak{m}\rangle(0 :_E \mathfrak{a}^n)$$

for every $n \in \mathbb{N}$. Since

$$\varinjlim(0 :_E \mathfrak{a}^n)/\langle\mathfrak{m}\rangle(0 :_E \mathfrak{a}^n) \simeq E/\sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_E \mathfrak{a}^n),$$

as is easily seen, and because $H^d_{\mathfrak{m}}(R) \simeq E$ for a $d$-dimensional Gorenstein ring $R$ this proves the claim in this case.

Now let $N$ denote an arbitrary finitely generated $R$-module over the Gorenstein ring such that $d = \dim_R N$. Therefore there are the isomorphisms

$$H^d_{\mathfrak{a}}(N) \simeq H^d_{\mathfrak{a}}(R)\otimes_R N \simeq (E/\sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_E \mathfrak{a}^n))\otimes_R N \simeq H^d_{\mathfrak{m}}(N)/\sum_{n\in\mathbb{N}}\langle\mathfrak{m}\rangle(0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^n).$$

This follows by 3.1 and because of the observation $H^d_{\mathfrak{m}}(R) \otimes_R N \simeq H^d_{\mathfrak{m}}(N)$.

Finally we reduce the case of an arbitrary local ring $(R, \mathfrak{m})$ to the previous situation. To this end put $R_1 := R/\operatorname{Ann}_R N$. Then there is the isomorphism

$$H^d_{\mathfrak{a}}(N) \otimes_{R_1} \hat{R}_1 \simeq H^d_{\mathfrak{a}\hat{R}_1}(\hat{N}).$$

By the Cohen Structure Theorem there exits a $d$-dimensional local Gorenstein ring $T$ such that $\hat{R}_1 \simeq T/\mathfrak{b}$ for a certain ideal $\mathfrak{b}$ of $T$. Then we get

$$H^d_{\mathfrak{c}\hat{R}_1}(\hat{N}) \simeq H^d_{\mathfrak{c}'}(\hat{N})$$

for any ideal $\mathfrak{c}$ of $R$ and its preimage $\mathfrak{c}'$ in $T$. Because $H^d_{\mathfrak{c}'}(\hat{N})$ is an Artinian $T$-module – as shown above – there is the isomorphism

$$H^d_{\mathfrak{c}}(N) \simeq H^d_{\mathfrak{c}'}(\hat{N})$$

for any ideal $\mathfrak{c}$ of $R$. Recall that $\hat{R}_1$ is a faithfully flat $R_1$-module. Therefore the reduction to the Gorenstein ring $T$ with $\dim T = \dim N = d$ is complete. This finishes the proof by the first part.

Finally let $N_1$ and $N_2$ be two $d$-dimensional $R$-modules. Then it is easy to see that one may use the same ring $R_1$ for both of $N_1$ and $N_2$ in order to proceed as in the previous consideration. Therefore in view of 3.1 the above isomorphism is functorial. □

Since $E = E(R/\mathfrak{m})$, the injective hull of the residue field, is an Artinian $R$-module it is easily seen that $\operatorname{Hom}_R(H^i_{\mathfrak{m}}(N), E)$ possesses the structure of an $\hat{R}$-module. Let $N$ be a finitely generated $R$-module. Since $H^i_{\mathfrak{m}}(N), i \in \mathbb{Z}$, is an Artinian $R$-module it follows by Matlis duality that $\operatorname{Hom}_R(H^i_{\mathfrak{m}}(N), E)$ is a finitely generated $\hat{R}$-module. We are particularly interested in

$$K(N) := \operatorname{Hom}_R(H^d_{\mathfrak{m}}(N), E),\ d = \dim_R N,$$

as an $\hat{R}$-module. Note that $K(N)$ is the completion of the canonical module $K_N$ of $N$ as introduced in [14, 3.1], provided $K_N$ does exist.

Furthermore let $0 = \cap_{i=1}^n Q_i$ denote a primary decomposition of the zero submodule of $K(N)$ as an $\hat{R}$-module.

**Corollary 3.3.** *With the previous notation the following statements are true:*

a) $\operatorname{Att}_{\hat{R}} H^d_{\mathfrak{a}}(N) = \{\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{N} \mid \dim \hat{R}/\mathfrak{p} = d$ *and* $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} = 0\}$.

b) $\operatorname{Hom}_R(H^d_{\mathfrak{a}}(N), E) \simeq \cap_{\dim \hat{R}/\mathfrak{a}\hat{R}+\mathfrak{p}_i>0} Q_i$, *where* $Q_i$ *denotes the* $\mathfrak{p}_i$*-primary component of the primary decomposition.*

*Proof.* First recall the well-known fact that

$$\operatorname{Att}_{\hat{R}} H^d_{\mathfrak{m}}(N) = \{\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{N} \mid \dim \hat{R}/\mathfrak{p} = d\},$$

see e.g. [2, 7.3.2]. By 3.2 and the minimal secondary representation of

$$\sum_{n\in\mathbb{N}} \langle\mathfrak{m}\rangle(0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^n),$$

shown in 2.8, the statement in a) results from the following observation:

Let $A = \sum_{i=1}^n A_i$ be minimal secondary representation of an $R$-module $A$, where $A_i$ is $\mathfrak{p}_i$-secondary. Set $B = \sum_{i=1}^m A_i$ for $0 \le m \le n$. Then $A/B = \sum_{i=m+1}^n (A_i + B)/B$ is a minimal secondary representation of $A/B$ such that $\operatorname{Att}_R A/B = \{\mathfrak{p}_{m+1}, \ldots, \mathfrak{p}_n\}$.

For the proof of b) let $H^d_{\mathfrak{m}}(N) = \sum_{i=1}^n A_i$ denote a minimal secondary representation of $H^d_{\mathfrak{m}}(N)$ as an $\hat{R}$-module. Because $K(N) = \operatorname{Hom}_R(H^d_{\mathfrak{m}}(N), E)$ it follows that

$$(0) = \cap_{i=1}^n \operatorname{Hom}_R(H^d_{\mathfrak{m}}(N)/A_i, E)$$

is a minimal primary decomposition of the zero submodule of $K(N)$ as an $\hat{R}$-module. Suppose that $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p}_i > 0$ for $0 \leq i \leq m$ while this does not hold for $m+1 \leq i \leq n$. By 2.8 it is known that

$$\sum_{n \in \mathbb{N}} \langle \mathfrak{m} \rangle (0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^n) = \sum_{i=1}^{m} A_i.$$

Then the short exact sequence $0 \to \sum_{i=1}^m A_i \to H^d_{\mathfrak{m}}(N) \to H^d_{\mathfrak{a}}(N) \to 0$ induces the isomorphim

$$\operatorname{Hom}_R(H^d_{\mathfrak{a}}(N), E) \simeq \cap_{i=1}^m \operatorname{Hom}_R(H^d_{\mathfrak{m}}(N)/A_i, E).$$

Therefore the statement results from the second uniqueness theorem for primary decompositions. Recall that $K(N)$ has no embedded associated prime ideals. □

The previous Corollary 3.3 has been shown by spectral sequence techniques and dualizing complexes in [15, (1.1)]. See also [3] for some particular cases.

As an application there is another approach to the Lichtenbaum-Hartshorne vanishing Theorem.

**Corollary 3.4.** *With the previous notation the following conditions are equivalent:*

(i) $H^d_{\mathfrak{a}}(N) = 0$.
(ii) $H^d_{\mathfrak{m}}(N) = \sum_{n \in \mathbb{N}} \langle \mathfrak{m} \rangle (0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^n)$.
(iii) *For any intger* $m \in \mathbb{N}$ *there exists an* $n = n(m) \in \mathbb{N}$ *such that*

$$0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^m \subseteq \langle \mathfrak{m} \rangle (0 :_{H^d_{\mathfrak{m}}(N)} \mathfrak{a}^n).$$

(iv) $\dim \hat{R}/\mathfrak{a}\hat{R} + \mathfrak{p} > 0$ *for all* $\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{N}$ *such that* $\dim \hat{R}/\mathfrak{p} = d$.
(v) *For any integer* $m \in \mathbb{N}$ *there exists an* $n = n(m) \in \mathbb{N}$ *such that*

$$\mathfrak{a}^n K(N) :_{K(N)} \langle \mathfrak{m} \rangle \subseteq \mathfrak{a}^m K(N).$$

The proof is clear by 3.2, 2.5, and 2.7. Note that the equivalence of (i) and (v) was shown by different arguments in [15, (1.1)].

## 4. Local cohomology and connectedness

In this section let $(R, \mathfrak{m})$ denote a local Noetherian ring. Let $\mathfrak{a}$ be an ideal of $R$. Then Hartshorne's connectedness result, see [6, Proposition 2.1], says that $\operatorname{Spec} R \setminus V(\mathfrak{a})$ is a connected subset of $\operatorname{Spec} R$ provided $\operatorname{grade}(\mathfrak{a}, R) > 1$. This result is shown with the aid of local cohomology, see 4.5 for a generalization to the case of modules. Therefore local cohomology modules reflect connectedness properties. This point of view was pursued further by the work of Faltings, Brodmann and Rung, and Hochster and Huneke, see [4], [1], and [8].

Here in the final section we want to apply some of our previous considerations in order to generalize some of these results to the case of a finitely generated $R$-module

$N$. To this end we put $\operatorname{Assh}_R N = \{\mathfrak{p} \in \operatorname{Ass}_R N \mid \dim R/\mathfrak{p} = \dim_R N\}$ for the set of highest-dimensional associated prime ideals.

**Lemma 4.1.** *Let $(R, \mathfrak{m})$ denote a local ring such that the nilradical of $\hat{R}$ consists of a single prime ideal $\mathfrak{p}$. Let $\mathfrak{b}, \mathfrak{c}$ denote two ideals of $R$ which are not $\mathfrak{m}$-primary. Suppose there exits a finitely generated $R$-module $N$ such that*

a) $d = \dim_R N > 1$,
b) $\operatorname{Assh}_{\hat{R}} \hat{N} = \{\mathfrak{p}\}$, *and*
c) $H^{d-1}_{\mathfrak{b}\cap\mathfrak{c}}(N) = 0$.

*Then it follows that $\mathfrak{b} + \mathfrak{c}$ is not an $\mathfrak{m}$-primary ideal.*

*Proof.* Suppose the contrary is true. That is, $\mathfrak{b} + \mathfrak{c}$ is an $\mathfrak{m}$-primary ideal. Then the Mayer-Vietoris sequence for local cohomology, see [16, 2.22], provides an exact sequence

$$H^{d-1}_{\mathfrak{b}\cap\mathfrak{c}}(N) \to H^d_{\mathfrak{m}}(N) \to H^d_{\mathfrak{b}}(N) \oplus H^d_{\mathfrak{c}}(N) \to H^d_{\mathfrak{b}\cap\mathfrak{c}}(N).$$

Because of the assumption in condition b) the Lichtenbaum-Hartshorne vanishing result, see 3.4, implies that $H^d_{\mathfrak{n}}(N) = 0$ for any ideal $\mathfrak{n}$ of $R$ such that $\dim R/\mathfrak{n} > 0$. Therefore because of the condition c) of the assumption the above short exact sequence yields that $H^d_{\mathfrak{m}}(N) = 0$, a contradiction to Grothendieck's non-vanishing result. Therefore $\mathfrak{b} + \mathfrak{c}$ is not an $\mathfrak{m}$-primary ideal, as required. □

The previous result has an application concerning the connectedness of $\operatorname{Supp} N/\mathfrak{a}N \setminus V(\mathfrak{m})$ for a certain ideal $\mathfrak{a}$.

**Corollary 4.2.** *Let $(R, \mathfrak{m})$ denote a local ring as in 4.1. Let $N$ be a finitely generated $R$-module such that $\operatorname{Assh}_{\hat{R}} \hat{N} = \{\mathfrak{p}\}, d = \dim_R N > 1$, and $H^i_{\mathfrak{a}}(N) = 0$ for $i = d-1, d$. Then $\operatorname{Supp} N/\mathfrak{a}N \setminus V(\mathfrak{m})$ is a connected subset of $\operatorname{Supp} N/\mathfrak{a}N$.*

*Proof.* Suppose that $\operatorname{Supp} N/\mathfrak{a}N \setminus V(\mathfrak{m})$ is not connected. Then there are ideals $\mathfrak{b}$ and $\mathfrak{c}$ of $R$ satisfying the following conditions

1) neither $\mathfrak{b}$ nor $\mathfrak{c}$ is $\mathfrak{m}$-primary,
2) $\operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R N) = \operatorname{Rad}(\mathfrak{b} \cap \mathfrak{c})$, and
3) $\mathfrak{b} + \mathfrak{c}$ is an $\mathfrak{m}$-primary ideal.

Because of the choice of $\mathfrak{b}$ and $\mathfrak{c}$ it follows that

$$H^i_{\mathfrak{b}\cap\mathfrak{c}}(N) \simeq H^i_{\mathfrak{a}+\operatorname{Ann}_R N}(N) \simeq H^i_{\mathfrak{a}}(N) = 0 \text{ for } i = d-1, d.$$

This provides a contradiction to 4.1. So the proof of 4.2 is complete. □

Now we are prepared in order to prove the main result of this section. It will generalize corresponding results of Hochster and Huneke, see [8], and Brodmann and Rung, see [1], that extend Faltings' original argument in [4].

**Theorem 4.3.** *Let $N$ denote a finitely generated $R$-module with $d = \dim_R N > 1$. Suppose that any minimal prime ideal of $\operatorname{Ass}_{\hat{R}} \hat{N}$ is of dimension $d$ and that $H^d_{\mathfrak{m}}(N)$ is an indecomposable $R$-module. Then*

$$\operatorname{Supp} N/\mathfrak{a}N \setminus V(\mathfrak{m})$$

*is connected provided $H^i_{\mathfrak{a}}(N) = 0$ for $i = d-1, d$.*

*Proof.* Suppose that the contrary is true. That is, there are ideals $\mathfrak{b}, \mathfrak{c}$ of $R$ such that $\mathfrak{b} + \mathfrak{c}$ is an $\mathfrak{m}$-primary ideal and

$$\operatorname{Rad} \mathfrak{b} \cap \mathfrak{c} = \operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R N),$$

while neither $\mathfrak{b}$ nor $\mathfrak{c}$ is an $\mathfrak{m}$-primary ideal. The Mayer-Vietoris sequence

$$H^{d-1}_{\mathfrak{a}}(N) \to H^d_{\mathfrak{m}}(N) \to H^d_{\mathfrak{b}}(N) \oplus H^d_{\mathfrak{c}}(N) \to H^d_{\mathfrak{a}}(N)$$

provides an isomorphism $H^d_{\mathfrak{m}}(N) \simeq H^d_{\mathfrak{b}}(N) \oplus H^d_{\mathfrak{c}}(N)$. By the assumption on the indecomposability of $H^d_{\mathfrak{m}}(N)$ one of the direct summands – say $H^d_{\mathfrak{b}}(N)$ – has to be zero, i.e. $H^d_{\mathfrak{m}}(N) \simeq H^d_{\mathfrak{c}}(N)$. By virtue of 3.3 a) – once applied with $\mathfrak{a} = \mathfrak{m}$ and a second time applied with $\mathfrak{a} = \mathfrak{c}$ – it follows that the set of minimal prime ideals $\operatorname{mAss}_{\hat{R}} \hat{N}$ of $\hat{N}$ is given by

$$\operatorname{mAss}_{\hat{R}} \hat{N} = \{\mathfrak{p} \in \operatorname{Ass}_{\hat{R}} \hat{N} \mid \dim \hat{R}/\mathfrak{p} = d \text{ and } \dim \hat{R}/\mathfrak{c}\hat{R} + \mathfrak{p} = 0\}.$$

Recall that by the assumption any minimal prime ideal is of dimension $d$. Hence $\mathfrak{c}\hat{R} + \mathfrak{p}$ is an $\mathfrak{m}\hat{R}$-primary ideal for all $\mathfrak{p} \in \operatorname{mAss}_{\hat{R}} \hat{N}$. Because of the equality

$$\operatorname{Rad} \cap_{\mathfrak{p} \in \operatorname{mAss}_{\hat{R}} \hat{N}} (\mathfrak{c}\hat{R} + \mathfrak{p}) = \operatorname{Rad}(\mathfrak{c}\hat{R})$$

it follows that $\operatorname{Rad} \mathfrak{c}\hat{R} = \mathfrak{m}\hat{R}$. Therefore $\operatorname{Rad} \mathfrak{c} = \mathfrak{m}$, in contradiction to the choice of $\mathfrak{b}$ and $\mathfrak{c}$. Hence $\operatorname{Supp} N/\mathfrak{a}N \setminus V(\mathfrak{m})$ is a connected subset of $\operatorname{Supp} N/\mathfrak{a}N$. ☐

**Remark 4.4.** Let $(R, \mathfrak{m})$ denote a complete local domain. Then one may identify the canonical module $K_R$ of $R$ with an ideal of $R$. In particular $K_R$ is indecomposable. Therefore, as mentioned by Hochster and Huneke, [8, Theorem 3.3] extends Faltings' original result. But although 4.2 and 4.3 extend Faltings' result and [8, Theorem 3.3] to finitely generated modules there are situations where 4.2 can be applied but 4.3 cannot.

To this end let $(R, \mathfrak{m})$ denote a complete local Gorenstein domain. Let $N$ be a finite direct sum of copies of $R$. Since the canonical module of $R$ is isomorphic to $R$ itself it follows that $K_N \simeq N$. Therefore, although $R$ and $N$ satisfy the assumptions of 4.2, the module $K_N$ is not indecomposable so that 4.3 cannot

It is an interesting problem to determine when $H^d_{\mathfrak{m}}(N)$ is an indecomposable $R$-module. For the case of $N = R$ this problem was solved by Hochster and Huneke, see [8, (3.6)]. In the case $N$ is equidimensional and satisfies the condition $S_2$ a necessary and sufficient condition is that $N$ itself has to be indecomposable. This follows easily by [16, 1.14].

Here we generalize these considerations to the case of an $R$-module $N$. To this end we need the following generalization of the connectedness result shown by Hartshorne in [6]. To this end let $N$ denote a finitely generated $R$-module, where $(R, \mathfrak{m})$ denotes a local Noetherian ring. For an arbitrary ideal $\mathfrak{a}$ of $R$ let $\operatorname{grade}(\mathfrak{a}, N)$ denote the grade of $N$ with respect to $\mathfrak{a}$. Recall the characterization

$$\operatorname{grade}(\mathfrak{a}, N) = \min\{n \in \mathbb{Z} \mid H^n_{\mathfrak{a}}(N) \neq 0\},$$

see [5] for the relation between local cohomology and grade.

**Lemma 4.5.** *Let $\mathfrak{a}$ denote an ideal of $(R, \mathfrak{m})$. Let $N$ denote a finitely generated indecomposable $R$-module such that* $\operatorname{grade}(\mathfrak{a}, N) > 1$. *Then the scheme* $\operatorname{Supp}_R N \setminus V(\mathfrak{a})$ *is connected.*

*Proof.* Because $\operatorname{grade}(\mathfrak{a}, N) > 1$ the vanishing of $H^i_{\mathfrak{a}}(N) = 0$ for $i = 0, 1$ follows. Suppose the contrary, i.e. $\operatorname{Supp}_R N \setminus V(\mathfrak{a})$ is not connected. Then there are two ideals $\mathfrak{b}, \mathfrak{c} \supseteq \operatorname{Ann}_R N$ satisfying the following properties:

1) $\operatorname{Rad}(\mathfrak{b} \cap \mathfrak{c}) = \operatorname{Rad}(\operatorname{Ann}_R N)$.
2) $\operatorname{Supp} N \setminus V(\mathfrak{b})$ and $\operatorname{Supp} N \setminus V(\mathfrak{c})$ are disjoint and non-empty subsets of $\operatorname{Supp}_R N$.
3) $\operatorname{Supp} N \setminus V(\mathfrak{a}) = \operatorname{Supp}(N \setminus V(\mathfrak{b})) \cup \operatorname{Supp}(N \setminus V(\mathfrak{c}))$.

Note that these conditions imply that $\operatorname{Rad}(\mathfrak{b} + \mathfrak{c}) = \operatorname{Rad}(\mathfrak{a} + \operatorname{Ann}_R N)$. Now recall that $H^i_{\mathfrak{a}}(N) \simeq H^i_{\mathfrak{a}+\operatorname{Ann}_R N}(N)$ for all $i \in \mathbb{Z}$. Then the first part of the Mayer-Vietoris sequence yields an exact sequence

$$0 \to H^0_{\mathfrak{a}}(N) \to H^0_{\mathfrak{b}}(N) \oplus H^0_{\mathfrak{c}}(N) \to H^0_{\mathfrak{b}\cap\mathfrak{c}}(N) \to H^1_{\mathfrak{a}}(N).$$

Because of the first of the above conditions we have $H^0_{\mathfrak{b}\cap\mathfrak{c}}(N) = N$. Whence the grade condition implies an isomorphism

$$H^0_{\mathfrak{b}}(N) \oplus H^0_{\mathfrak{c}}(N) \simeq N.$$

But by the assumption $N$ is indecomposable, so either $H^0_{\mathfrak{b}}(N) = 0$ and $H^0_{\mathfrak{c}}(N) \simeq N$ or $H^0_{\mathfrak{b}}(N) \simeq N$ and $H^0_{\mathfrak{c}}(N) = 0$. But this means that either $\operatorname{Rad} \mathfrak{b} = \operatorname{Rad} \operatorname{Ann}_R N$ or $\operatorname{Rad} \mathfrak{c} = \operatorname{Rad} \operatorname{Ann}_R N$. Therefore we have arrived at a contradiction, so $\operatorname{Supp}_R N \setminus V(\mathfrak{a})$ is connected. ☐

In the following we need a definition about connectedness first introduced by Hartshorne, see [6]. In fact in 4.8 we shall relate this purely topological notion to a certain algebraic property.

**Definition 4.6.** Let $X$ denote a subset of the spectrum $\operatorname{Spec} R$ of a commutative Noetherian ring $R$. The set $X$ is called connected in codimension one provided for any two irreducible components $Y$ and $Z$ of $X$ there exists a family of irreducible components $Y = X_0, \ldots, X_i, \ldots, X_n = Z$ such that $\operatorname{codim}_X(X_{i-1} \cap X_i) \leq 1$ for all $1 \leq i \leq n$.

For an $R$-module $N$ we say that $N$ is indecomposable in codimension 1 whenever $N_{\mathfrak{p}}$ is an indecomposable $R_{\mathfrak{p}}$-module for all prime ideals $\mathfrak{p} \in \operatorname{Supp}_R N$ such that $\dim N_{\mathfrak{p}} \leq 1$. Here indecomposability means that the corresponding module is not the non-trivial direct sum of two modules. The following lemma relates the indecomposability to connectedness properties.

**Lemma 4.7.** *Let $(R, \mathfrak{m})$ be a local ring. Let $N$ denote a finitely generated $R$-module. Suppose that $N$ satisfies the following conditions:*

a) *$N$ is equidimensional and unmixed.*
b) *$\operatorname{Supp}_R N$ is connected in codimension 1.*
c) *$N$ is indecomposable in codimension 1.*

*Then $N$ itself is indecomposable.*

*Proof.* Suppose that $N = U \oplus V$ for two non-trivial $R$-modules $U$ and $V$. Let $X = \operatorname{Supp}_R U$ and $Y = \operatorname{Supp}_R V$. Because $\operatorname{Supp}_R N = X \cup Y$ the assumption implies that $U$ and $V$ are both equidimensional and unmixed. Now choose two minimal prime ideals $\mathfrak{p} \in X$ and $\mathfrak{q} \in Y$. Then there is a chain

$$\mathfrak{p} = \mathfrak{p}_0, \mathfrak{p}_1, \ldots, \mathfrak{p}_s = \mathfrak{q}$$

of minimal prime ideals of $\operatorname{Supp}_R N$ and a chain $\mathfrak{P}_1, \ldots \mathfrak{P}_s$ of prime ideals of codimension $\leq 1$ in $\operatorname{Supp}_R N$ such that

$$\mathfrak{p}_{i-1} + \mathfrak{p}_i \subseteq \mathfrak{P}_i \text{ for } i = 1, \ldots, s.$$

Because of the indecomposability in codimension 1 it follows $\mathfrak{p} = \mathfrak{p}_0 \in X$ and $\mathfrak{p}_0 \notin Y$. Hence $\mathfrak{P}_1 \in X$ and $\mathfrak{P}_1 \notin Y$ by the indecomposability in codimension 1. Therefore $\mathfrak{p}_1 \in X$ and $\mathfrak{p}_1 \notin Y$. By iterating this argument $s$ times it follows that $\mathfrak{p}_s = \mathfrak{q} \notin Y$, contracting the choice of $\mathfrak{q} \in Y$. □

In the following we shall use the notion of connectedness in order to characterize when the local cohomology module $H^d_{\mathfrak{m}}(N), d = \dim_R N$, is an indecomposable $R$-module. In the case $R$ is a factor ring of a Gorenstein ring let $K_N$ denote the canonical module of $N$ in the sense of [14, 3.1]. Then $K(N) \simeq K_N \otimes_R \hat{R}$.

**Theorem 4.8.** *Let $\mathfrak{a}$ denote an ideal of a local ring $(R, \mathfrak{m})$ which is a factor ring of a Gorenstein local ring. Let $N$ be a finitely generated equidimensional, unmixed $R$-module with $d = \dim_R N > 1$. Suppose that $N$ is indecomposable in codimension 1. Then the following conditions are equivalent:*

(i) *$H^d_{\mathfrak{m}}(N)$ is an indecomposable $R$-module.*
(ii) *$K_N$ is an indecomposable $R$-module.*

(iii) $\operatorname{Supp}_R N \setminus V(\mathfrak{a})$ *is connected for any ideal* $\mathfrak{a}$ *of* $R$ *such that*

$$\operatorname{height}(\mathfrak{a} + \operatorname{Ann}_R N / \operatorname{Ann}_R N) > 1.$$

(iv) $\operatorname{Supp}_R N$ *is connected in codimension* 1.

*Proof.* First we observe that the equivalence of (i) and (ii) is obviously true.

Now we show the implication (ii) $\Rightarrow$ (iii). In this case $K_N$ is the canonical module of $N$ as defined in [14, 3.1]. Then $\operatorname{Supp}_R N = \operatorname{Supp}_R K_N$ and $K_N$ satisfies the condition $S_2$, see [14, 3.1.1]. That is

$$\operatorname{depth}_{R_\mathfrak{p}} (K_N)_\mathfrak{p} \geq \min\{2, \dim_{R_\mathfrak{p}} (K_N)_\mathfrak{p}\} \text{ for all } \mathfrak{p} \in \operatorname{Supp} K_N.$$

It is easily seen that this implies $\operatorname{grade}(\mathfrak{a}, K_N) > 1$ for any ideal $\mathfrak{a}$ such that

$$\operatorname{height}(\mathfrak{a} + \operatorname{Ann}_R N / \operatorname{Ann}_R N) > 1.$$

Recall that $\operatorname{Rad} \operatorname{Ann}_R N = \operatorname{Rad} \operatorname{Ann}_R K_N$. By virtue of 4.5 this proves (iii).

Next we have to mention that the equivalence of (iii) and (iv) is a particular case of [6, Proposition 1.1].

In order to complete the proof we show the implication (iv) $\Rightarrow$ (ii). To this end let us apply Lemma 4.7. Note that $K_N$ is equidimensional and unmixed. Moreover $\operatorname{Supp}_R K_N = \operatorname{Supp}_R N$. Therefore it will be enough to prove that $K_N$ is indecomposable in codimension 1. First note that since $R$ is a quotient of a Gorenstein ring and since $N$ is equidimensional it follows that

$$\dim N = \dim N_\mathfrak{p} + \dim R/\mathfrak{p} \text{ for all } \mathfrak{p} \in \operatorname{Supp}_R N.$$

Therefore $K_{N_\mathfrak{p}} \simeq K_N \otimes_R R_\mathfrak{p}$, for all $\mathfrak{p} \in \operatorname{Supp}_R N$, see [16, 1.9]. Second fix a prime ideal $\mathfrak{p} \in \operatorname{Supp}_R N$ such that $\dim K_N \otimes_R R_\mathfrak{p} \leq 1$. Suppose that $K_{N_\mathfrak{p}} \simeq K_N \otimes_R R_\mathfrak{p}$ is decomposable, i.e. $K_{N_\mathfrak{p}} = U \oplus V$ for two non-trivial $R_\mathfrak{p}$-modules $U$ and $V$. Then it turns out that

$$\dim U = \dim V = \dim K_{N_\mathfrak{p}} = \dim N_\mathfrak{p}.$$

Moreover $N_\mathfrak{p}$ is a Cohen-Macaulay module over $R_\mathfrak{p}$. By [16, 1.14] we therefore have that

$$N_\mathfrak{p} \simeq K_{K_{N_\mathfrak{p}}} \simeq K_U \oplus K_V.$$

Since both – $K_U$ and $K_V$ – are non-trivial this provides a contradiction to the indecomposability of $N_\mathfrak{p}$. So the proof is finished. □

Of course in the case of $N = R$, the underlying ring, $R$ is indecomposable in codimension 1. Moreover, one might factor out from $R$ the ideal of largest dimension less than $\dim R$ such that the resulting homomorphic image is equidimensional and unmixed. Then one sees that 4.8 is a generalization of [8, 3.6].

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