æ CURVES OF DEGREE $r+2$ IN $\mathbb{P}^r$: COHOMOLOGICAL, GEOMETRIC AND HOMOLOGICAL ASPECTS

Markus Brodmann *and* **Peter Schenzel**$^{*)}$

Abstract: *Let $\mathcal{C} \subseteq \mathbb{P}^r_K$ be a non- degenerate projective curve of degree $r+2$, where $r \geq 3$. By means of the Hartshorne-Rao module of $\mathcal{C}$ we distinguish 4 different possible cases (and an exceptional case which only appears if $r = 3$). In any case $\mathcal{C}$ is obtained either by means of an embedding of an arbitrary smooth curve $\mathcal{C}_0$ of genus 2, or by projecting an elliptic normal curve from a point or by projecting a rational normal curve from a line. Finally, the possible minimal free resolutions of the homogeneous coordinate ring of $\mathcal{C}$ are studied.*

1. INTRODUCTION

Let K be an algebraically closed field and let $X \subseteq \mathbb{P}^r_K$ be a non-degenerate projective variety. Then, the degree and the codimension of X satisfy the well known inequality $\deg(X) \geq$ codim $(X) + 1$. If X is of *minimal degree*, e.g. if $\deg(X) = \operatorname{codim}(X) + 1$, the structure of X is well known (cf. [Ei- Ha], [Ha$_2$, (3.10)]). In particular, if X is a curve of minimal degree, it must be a rational normal curve. Moreover, a surface of minimal degree is either a cone over a rational normal curve, the Veronese surface in $\mathbb{P}^5$ or a rational normal scroll (s. [Be, Ch. IV]).

If $\deg(X) = \operatorname{codim}(X)+2$, still rather much is known on the structure of X, in particular from the homological point of view, (s. [Ho-St-V]). Less is known in this case on the geometric aspect unless X is a curve or a smooth surface: If X is a non-singular curve it is either an elliptic normal curve or obtained by projecting a rational normal curve from a point (for details see (4.7)B)). The case of (complex) smooth surfaces with $\operatorname{reg}(X) = \operatorname{codim}(X) + 2$ is partially covered by the so called Del Pezzo classification (cf. [DP]). Even as far as $\deg(X) \leq \operatorname{codim}(X) + 5$ rather much is known on the structure of X, provided X is a complex smooth surface (s. [E]).

These observations gave the motivation for the present paper, whose aim is to study in detail all curves X with $\deg(X) = \operatorname{codim}(X) + 3$.

So, let $\mathcal{C} \subseteq \mathbb{P}^r_K$ be a non-degenerate curve of degree $r + 2$, where $r \geq 3$. Let $I \subseteq K[\mathbf{x}_0, \ldots, \mathbf{x}_r] =: S$ be the defining ideal of $\mathcal{C}$ and let $A = S/I$ be the homogeneous coordinate ring of $\mathcal{C}$.

We attack our problem from the cohomological side by looking at the *Hartshorne-Rao module* of $\mathcal{C}$, e.g. the

---

$^{*)}$ Second author partially supported by Swiss National Science Foundation; Projects No. 20-52762.97 and 21-36526.92. 2000 *Mathematical subject classification:* primary 14H45; secondary 13D02. *Key words and phrases.* Projective curves, Hartshorne-Rao module, rational normal curve, elliptic normal curve, surface of minimal degree, Betti numbers, Castelnuovo regularity first local cohomology module $H^1(A)$ of A with respect to the irrelevant maximal ideal A_+ of A. It turns out, that only four different types of Hartshorne-Rao modules occur — plus an additional type which appears only if $r = 3$, (s. (3.9), (3.10), (3.11)). We take these 4(+1) *cohomology types* as the basis for a first classification of our curves, considering four possible cases I, II, III, IV and an exceptional case IV$''$.

It is important to notice that in all cases the Hartshorne-Rao module $H^1(A)$ is generated in degree 1 (s. (3.5)a)). This means that the global transform $D(A) = \oplus_{n\in\mathbb{Z}}\Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}}(n))$ of $\mathcal{C}$ is generated as a K-algebra by homogeneous elements of degree 1 and thus is the homogeneous coordinate ring of a non-degenerate arithmetically Cohen-Macaulay curve $\mathcal{C}' \subseteq \mathbb{P}^{r'}_K$ with $r' = r + h^1(A)_1$, where $h^1(A)_1$ is the dimension of the first homogeneous part $H^1(A)_1$ of $H^1(A)$. In particular, $\mathcal{C}$ is obtained by projecting $\mathcal{C}'$ from a linear subspace $P = \mathbb{P}^{r'-r-1}_K \subseteq \mathbb{P}^{r'}_K$ which does not meet the secants of $\mathcal{C}'$, (s. (4.1)). If $r' > r$, the curve $\mathcal{C}'$ satisfies $\deg(\mathcal{C}') = r + 2 \leq \operatorname{codim}(\mathcal{C}') + 2$ and hence is well understood. In the smooth case this already leads to a satisfactory geometric description of $\mathcal{C}$ (s. (4.3)), either as an embedding of an arbitrary smooth curve $\mathcal{C}_0$ of genus 2, or of a nonsingular projection of an elliptic normal curve from a point or a nonsingular projection of a normal rational curve from a line. The latter principle of construction completely recovers the cases III, IV and IV$''$. In the cases I and II, $\mathcal{C}$ may be singular. So, the case I splits up into four subcases I$_0$, I$_1$, I$_2$ and I$'_2$ according to whether $\mathcal{C}$ is smooth, has a double point, a triple point or two double points. The case II splits up into two subcases II$_0$ and II$_1$, according to whether $\mathcal{C}$ is smooth or has a double point. As the normalization of the associated curve $\mathcal{C}'$ may be realized by a projection of an arithmetically normal curve, we may describe $\mathcal{C}$ as either an appropriate (singular) projection of an elliptic normal curve from a point or of a rational normal curve from a line (s. (4.6)).

Chapters 5 and 6 of the present paper are devoted to the homological aspect of our curves – the study of their Betti modules. We attack this problem in two steps. The first step consists of a complete calculation of the Betti modules of the S-module $D(A)$ in all possible cases (s. (5.1)a), (5.3)a),b)). To do so, we first write $D(A)$ as a homomorphic image of a polynomial ring $S' = K[\mathbf{x}_0, \ldots, \mathbf{x}_{r'}]$ and determine the Betti-modules of $D(A)$ over S. In the case I, we can apply a result of Green-Lazarsfeld [G-L] (resp. Nagel [N]) on $r + 2$ *points in semiuniform position in* $\mathbb{P}^{r-1}_K$ in order to determine the requested modules. In the remaining cases $D(A)$ is either an *extremal Gorenstein ring* or an extremal *Cohen-Macaulay ring of size 2* and so we get what we want by [S$_1$], (s. (5.1)a), b), c)). To study the Betti modules of $D(A)$ over S, we essentially have to compare homologies of the two Koszul complexes $\mathbb{K}_\bullet\left(\mathbf{x}_0, \ldots, \mathbf{x}_{r'}; D(A)\right)$ and $\mathbb{K}_\bullet\left(\mathbf{x}_0, \ldots, \mathbf{x}_r; D(A)\right)$ of $D(A)$. This may in fact be done by a repeated use of the comparison sequence for Koszul homologies found in [Bru-He] (s. also (5.2)) and by whatching carefully what happens in the single homogeneous parts of the occuring homology modules.

In Chapter 6, the Betti modules of the S-module A are studied in the cases II, III and IV. Our basic idea is to use the natural exact sequence

$$0 \to A \to D(A) \to H^1(A) \to 0$$

in conjunction with our knowledge of the Betti modules of $D(A)$ and the fact that we know the structure of the S-modules $H^1(A)$. Another tool is to intersect $\mathcal{C}$ with a generic hyperplane $\mathbb{P}^{r-1}_K \subseteq \mathbb{P}^r_K$ and to exploit the minimal free resolution of the vanishing ideal of $r + 2$ points in "general position" in the hyperplane $\mathbb{P}^{r-1}_K$.

By a theorem of M. Green [G] the structure of the Betti module $\operatorname{Tor}^S_{r-2}(K, A)$ is closely related to the fact whether the curve $\mathcal{C}$ is contained in a surface $X \subseteq \mathbb{P}^r_K$ of minimal degree. Much emphasis is given to this link between geometry and homology in chapter 6, and for this reason we begin this chapter with the needed preliminaries on surfaces of minimal degree. As a basic reference for such surfaces we use Beauville [Be], where all the necessary results are established over the complex numbers. For the case of arbitrary algebraically closed ground fields the reader should consult the corresponding appendix in [Be]. At first instance we shall see, that in the case I, the curve $\mathcal{C}$ always is contained in a surface $X \subseteq \mathbb{P}^r_K$ of minimal degree and that X is non-singular if $r \geq 5$, (cf. (6.3)A)).

In the case II and if $r \geq 5$, $\mathcal{C}$ is never contained in a surface of minimal degree (cf. (6.3)B)) and moreover, the defining ideal $I \subseteq S$ of $\mathcal{C}$ is generated by quadrics and one cubic (cf. (6.4)). In (6.6) we use this to approximate the Betti modules $\operatorname{Tor}^S_i(K, A)$ of $\mathcal{C}$ in the case II. In (6.7) we give a few examples calculated by means of *Macaylay* [Ba-St] and illustrating different phenomena that may occur in the case II.

In (6.9) and (6.10) we approximate the Betti modules of $\mathcal{C}$ for the case III and list a few examples which go under this case. In (6.12) and (6.13) we do the same for the case IV. In particular in the cases III and IV it may occur for each $r \geq 4$ that $\mathcal{C}$ is contained in a (non-singular) surface $X \subseteq \mathbb{P}^r_K$ of minimal degree.

In the case IV, the defining ideal of $\mathcal{C}$ has the form $I = (J, Q)$, where $J \subseteq S$ is generated by quadrics and cubics and where $Q \in S$ is a quartic. We shall see that $(J :_S Q)$ defines a quadrisecant line to $\mathcal{C}$, the line which must exist by Gruson-Lazarsfeld-Peskine [Gr-L-P]. Moreover it will turn out that S/J is a Cohen-Macaulay ring of dimension 2, (cf. (6.14)).

We thank D. Eisenbud and C. Peskine for their hints and helpful comments during the preparation of this paper.

2. PRELIMINARIES

We first fix a few notations, which we shall use later without further comment. If $R = \oplus_{n\geq 0} R_n$ is a positively graded ring, we write Mod^*_R for the category of graded R-modules. If M is a graded R-module, and if $n \in \mathbb{Z}$, we write M_n for the n-th homogeneous part of M, so that as an R_0-module M may be written in the form $M = \bigoplus_{n\in\mathbb{Z}} M_n$. If $h : M \to N$ is a homomorphism of graded R-modules, $h_n : M_n \to N_n$ shall denote the n-th homogeneous part of h. If M is a graded R-module and if r is an integer, we write $M(r)$ for the r-shift of M, so that $M(r)_n = M_{n+r}$ for all $n \in \mathbb{Z}$. The same notation is used to denote shifts of homomorphisms of graded R-modules.

By R_+ we denote the irrelevant ideal $\bigoplus_{n>0} R_n$ of R. For $i \in \mathbb{N}_0$ let H^i denote the i-th local cohomology functor $H^i_{R_+} = \varinjlim_{\mathrm{n}} \mathrm{Ext}^i_R\,(R/(R_+)^n, \bullet)$ with support R_+, which is defined in the category Mod_R of all R-modules. Let $D : \mathrm{Mod}_R \to \mathrm{Mod}_R$ the functor $D_{R_+} = \varinjlim_{\mathrm{n}} \mathrm{Hom}_R((R_+)^n, \bullet)$ of R_+-transforms.

(2.1) Remark. A) (We refer to [Br-Sh]). The functors H^i and D convert graded modules into graded modules and homogeneous homomorphisms into homogeneous homomorphisms, so that they induce functors $H^i, D : \mathrm{Mod}^*_R \to \mathrm{Mod}^*_R$. Moreover, both functors commute with shifting, so that for a graded R-module M we may write $H^i(M(r)) \cong H^i(M)(r)$ and $D(M(r)) \cong D(M)(r)\,,\ (i \in \mathbb{N}_0,\ r \in \mathbb{Z})$.

B) If R is noetherian, and if M is a finitely generated graded R-module, then the n-th homogeneous parts $H^i(M)_n$ and $D(M)_n$ of the graded R-modules $H^i(M)$ and $D(M)$ are both finitely generated R_0-modules and moreover we have $H^i(M)_n = 0$ for all $n \gg 0$. So, if R_0 is artinian and if R is of finite type over R_0 and if M is a finitely generated graded R-module, the R_0-modules $H^i(M)_n$ and $D(M)_n$ are of finite length. In this case, the lengths of the modules $H^i(M)_n$ and $D(M)_n$ are denoted respectively by $h^i(M)_n$ and $d(M)_n$. •

If a positively graded ring $R = \bigoplus_{n\geq 0} R_n$ is generated over its 0-th homogeneous part R_0 by homogeneous elements of degree 1, we say that R is a *homogeneous* R_0-algebra.

(2.2) Remark. Let $R = R_0[R_1] = \bigoplus_{n\geq 0} R_n$ be a homogeneous R_0-algebra and let $X = \mathrm{Proj}(R)$. If M is a graded R-module, let $\tilde{M}$ denote the quasicoherent sheaf of $\mathcal{O}_X$-modules which is induced by M. If $\mathcal{O}_X(1) := R(1)^\sim$ denotes the induced twisting sheaf on X, we have natural isomorphisms

$$M(r)^\sim \cong \tilde{M}(r) := \tilde{M} \otimes \mathcal{O}_X(1)^{\otimes r} \quad (r \in \mathbb{Z})\ .$$

Moreover, if $H^i(X, \mathcal{F})$ denotes the i-th Serre-cohomology-module of X with coefficients in a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, the *Serre-Grothendieck-Correspondence* yields natural isomorphisms of graded R-modules (s. [Br-Sh])

$$\bigoplus_{n\in\mathbb{Z}} H^i\big(X, \tilde{M}(n)\big) \cong H^{i+1}(M), \quad (i \geq 1);$$
$$\bigoplus_{n\in\mathbb{Z}} H^0\big(X, \tilde{M}(n)\big) \cong D(M).$$

•

We now give a first auxiliary result on graded modules over homogeneous rings.

(2.3) Lemma. *Let R_0 be an artinian ring, let R be a homogeneous R_0-algebra of finite type and let M be a finitely generated and graded R-module of Krull dimension 1. Then:*

a) *For all $n \in \mathbb{Z}$, $d(M)_n$ equals the length of the coherent sheaf $\tilde{M}$ which is induced on Proj(R) by M.*

b) *If M is generated by homogeneous elements of degree $\leq t$, then*

$$h^1(M)_{n+1} \leq \max\{0, h^1(M)_n - 1\} \quad \textit{for all} \quad n \geq t \ .$$

Proof: Statement a) follows immediately from the Serre-Grothendieck correspondence (see (2.2)).

To prove statement b), we first denote by $\mathfrak{m}_1, \ldots, \mathfrak{m}_p$ the different maximal ideals of R_0. Let T be an indeterminate and let $S := R_0[T] \setminus \bigcup_{i=1}^{p} \mathfrak{m}_i R_0[T]$. Then $\overline{R}_0 := S^{-1}(R_0[T])$ is faithfully flat over R_0. Moreover $\overline{R}_0$ is artinian with the maximal ideals $\overline{\mathfrak{m}}_i := \mathfrak{m}_i \overline{R}_0$ $(i = 1, \ldots, p)$ and we have $\overline{R}_0/\overline{\mathfrak{m}}_i \cong (R/\mathfrak{m}_i)(T)$ for $i = 1, \ldots, p$. In view of the flat base change property of local cohomology we now may replace R by the homogeneous $\overline{R}_0$-algebra $\overline{R}_0 \otimes_{R_0} R$ and M by the graded $\overline{R}_0 \otimes_{R_0} R$-module $\overline{R}_0 \otimes_{R_0} M$. We thus may assume that the residue fields $R_0/\mathfrak{m}_i$ of R_0 are infinite. Moreover we may replace M by $M/H^0(M)$ and thus assume that R_+ contains M-regular elements. As R_0 has infinite residue fields, we then find an M-regular element $\ell \in R_1$.

Now, let $n \geq t$ and assume that $h^1(M)_{n+1} > \max\{0, h^1(M)_n - 1\}$. Applying cohomology to the exact sequence $0 \to M \xrightarrow{\ell} M(1) \to M/\ell M(1) \to 0$ and observing that $\dim(M/\ell M(1)) = 0$ we get an epimorphism $H^1(M)_n \xrightarrow{\ell} H^1(M)_{n+1}$. So, by our assumption we obtain

$$h^1(M)_{n+1} = h^1(M)_n > 0 \ .$$

Now, let $d :=$ length $(\tilde{M})$. Then, statement a) and the short exact sequence $0 \to M \to D(M) \to H^1(M) \to 0$ show that the R_0-modules M_n and M_{n+1} are both of the same length $d' < d$. In particular, the multiplication map $\ell : M_n \to M_{n+1}$ becomes bijective. As M is generated in degree $t \leq n$ and as R is homogeneous, each element $m \in M_{n+2}$ may be written as $m = \sum_{i=1}^{r} x_i u_i$, with appropriately choosen elements $x_1, \ldots, x_r \in R_1$ and $u_1, \ldots, u_r \in M_{n+1}$. The surjectivity of the map $\ell : M_n \to M_{n+1}$ allows to write $u_i = \ell v_i$, with $v_i \in M_n$, and so $m = \sum_{i=1}^{r} x_i u_i = \ell \sum_{i=1}^{r} x_i v_i$ shows, that the map $\ell : M_{n+1} \to M_{n+2}$ is surjective too. Going on like this, we see that all the maps $\ell : M_{n+k} \to M_{n+k+1}$ $(k \in \mathbb{N}_0)$ are surjective, thus bijective. Therefore, all the modules M_{n+k} with $k \in \mathbb{N}_0$ are of the same length $d' < d$, and we get the contradiction that $h^1(M)_{n+k} = d - d' > 0$ for all $k \in \mathbb{N}_0$, (s. (2.1) B)). ■

We now recall a few facts on point sets in "general position" in projective spaces. So, let K be an algebraically closed field and let $s \in \mathbb{N}$. By $\mathbb{P}^s_K$ we denote the projective s-space over K. Let $p_1, \ldots, p_d$ be d different closed points of $\mathbb{P}^s_K$. We say that these d points are in *semi-uniform position* (s. [B]), if the following two conditions are satisfied:

- The set $P = \{p_1, \ldots, p_d\}$ spans the whole space $\mathbb{P}^s_K$.
- If $L_1, L_2 \subseteq \mathbb{P}^s_K$ are linear subspaces of the same dimension such that L_i is spanned by $P \cap L_i$ for $i = 1, 2$, then the two sets $P \cap L_1$ and $P \cap L_2$ are of the same cardinality.

Correspondingly, a *scheme of d points in semi-uniform position* in $\mathbb{P}^s_K$ is a closed, reduced subscheme of dimension 0 in $\mathbb{P}^s_K$, whose underlying set consists of d points in semi-uniform position in $\mathbb{P}^s_K$.

(2.4) Lemma. *Let K be an algebraically closed field and let R be a graded homomorphic image of the polynomial ring $S := K[\mathbf{x}_0, \ldots, \mathbf{x}_s]$ such that Proj(R) is a scheme of d points in semi-uniform position in $\mathbb{P}^s_K :=$ Proj(S). Then:*

a) $h^1(R)_n \leq \max\{0, d - 1 - ns\}$ *for all* $n \in \mathbb{N}_0$.

b) *If $s \geq 3$ and if $d = s+3$, then the graded R-module $H^0(R) \subseteq R$ is generated by homogeneous elements of degree 2.*

Proof: a) The homogeneous coordinate ring of $X := \mathrm{Proj}(R)$ is given by $\overline{R} := R/H^0(R)$. Let $\overline{r}_n$ denote the dimension of the K-vectorspace $\overline{R}_n$. Then, by [N$_1$, 2.1.2.2] (cf. also [B]) we have $\overline{r}_n \geq \min\{d, ns+1\}$ for all $n \geq 0$. But in view of the natural graded exact sequence $0 \to \overline{R} \to D(R) \to H^1(R) \to 0$ and in view of (2.3)a) (applied with $M = R$) we have $h^1(R)_n = d - \overline{r}_n$. Altogether, this proves our claim.

b) By the "N_p-Theoreme" of Green-Lazarsfeld [G-L] (see also [N$_2$, Theorem 1]) we know that the homogeneous vanishing ideal $I \subseteq S$ of a reduced scheme $X \subseteq \mathbb{P}^s_K$ of $s+3$ closed points is generated by homogeneous polynomials of degree 2 if and only if no 3 points of X lie on a line $L \subseteq \mathbb{P}^s_K$ and no 5 points of X lie on a plane $E \subseteq \mathbb{P}^s_K$. As the points of X are in semi-uniform position in $\mathbb{P}^s_K$ and as $s > 2$, these latter coincidence conditions are indeed satisfied, and thus the vanishing ideal I of X is generated by quadrics. By means of epimorphism in the first column of the canonical diagram of graded homorphisms

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
0 \arrow[r] & I \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & S/I \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H^0(R) \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & \overline{R} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

we thus get our claim. ■

Statement (2.4) b) concerns $s+3$ points in $\mathbb{P}^s_K$. We now will prove another auxiliary result which concerns this case and which will be used later (see proof of (3.5)).

(2.5) Lemma. *Let K and $S = K[\mathbf{x}_0, \ldots, \mathbf{x}_s]$ be as in (2.4) and let R be a graded homomorphic image of the polynomial ring S such that Proj(R) is a reduced scheme of $s+3$ points. Assume that $h^1(R)_2 = 1$. Then, the graded R-module $H^0(R)$ is generated by homogeneous elements of degree 2 and one homogeneous element of degree 4.*

Proof: We write again $\overline{R}$ for the homogeneous coordinate ring $R/H^0(R)$ of $X := \mathrm{Proj}(R)$ and I for the homogeneous vanishing ideal of X in S. As $h^1(R)_2 = 1$ the natural short exact sequence $0 \to \overline{R} \to D(R) \to H^1(R) \to 0$ and (2.3)a) (applied with $M = R$) show that $\overline{R}_2$ is a K-vector space of dimension $s+2$. So [N$_2$, Theorem 2] shows that the vanishing ideal I may be generated by $\binom{s+1}{2} - 1$ quadrics and one quartic. Now we may conclude as in the proof of (2.4)b). ■

(2.6) Remark. A) Let $\mathcal{C} \subseteq \mathbb{P}^r_K$ a closed, one dimensional, integral, non-degeneratedly embedded subscheme of degree d. Then, by the "semi-uniform position lemma" (s. [B], [Ha$_1$], [N$_1$, 2.1.1.7]) there is a hyperplane $H \subseteq \mathbb{P}^r_K$ such that $\mathcal{C} \cap H$ is a scheme of d points in semi-uniform position in $H = \mathbb{P}^{r-1}_K$.

This last fact holds in fact true for a generic hyperplane H in $\mathbb{P}^r_K$. Therefore an element ℓ of the homogeneous coordinate ring $A = K \oplus A_1 \oplus \ldots$ of $\mathcal{C}$ is called a *generic linear form*, if it is homogeneous of degree 1 and if $\mathrm{Proj}(A/\ell A) = \mathcal{C} \cap H$ is a subscheme of d points in semi-uniform position in the hyperplane $H \subseteq \mathbb{P}^r_K$ defined by ℓ.

B) Later, we shall use the following observations on sets of points in semi-uniform position: let $s \geq 2$ and let $X \subseteq \mathbb{P}^s_K$ be a scheme of $s+3$ points in semi-uniform position. Then, if S denotes the polynomial ring $K[\mathbf{x}_0, \ldots, \mathbf{x}_s]$, the defining ideal $I \subseteq S$ of X has a minimal graded free resolution of the form

$$0 \to F_s \to F_{s-1} \to \ldots \to F_1 \to I \to 0$$

with

$$F_i \;\cong\; \begin{cases} S^{a_i} & \text{for } 1 \le i \le S-2\ , \\ S^s(-s-1) \oplus S^{s-1}(-S) & \text{for } i = S-1\ , \\ S^2(-S-2) & \text{for } i = S\ , \end{cases}$$

where

$$a_i \;=\; i\binom{S+1}{i+1} - 2\binom{S}{i-1} \quad \text{for} \quad 1 \le i \le S-2\ ,$$

(see [G-L] and [N_2, Theorem 2]). •

(2.7) Remark. We now recall an argument which will be used frequently and which may be found in [Br_1, 3.4]. Let $R = K \oplus R_1 \oplus R_2 \oplus \ldots$ be a finitely generated graded algebra over an algebraically closed field K, let M be a finitely generated graded R-module, let $i \in \mathbb{N}_0$ and let $n \in \mathbb{Z}$. Moreover let $L \subseteq R_1$ be a K-linear subspace of dimension $r+1$ ($r \in \mathbb{N}_0$) such that the multiplication map $\ell : H^i(M)_n \to H^i(M)_{n+1}$ is surjective (resp. injective) for each $\ell \in L \setminus \{0\}$. Then $h^i(M)_{n+1} \le \max\{0, h^i(M)_n - r\}$ $\big($resp. $h^i(M)_n \le \max\{0, h^i(M)_{n+1} - r\}\big)$. •

Now, let $R = K \oplus R_1 \oplus \ldots$ be a homogeneous and finitely generated K-algebra, where K is a field. Let M be a finitely generated and graded R-module of dimension d. As usually, we write K for the graded R-module $R/R_{>0}$ and define the *type* of a finitely generated graded R-module M by $\tau(M) = \tau_R(M) := \dim_K \operatorname{Ext}^d_R(K, M)$.

(2.8) Remark. A) Keep the above hypotheses and notations. If $x \in R_+$ is a homogeneous M-regular element, we have $\tau_R(M/xM) = \tau_R(M)$. So, if M is a CM-module and if $x_1, \ldots, x_d$ is a homogeneous M-sequence in R_+, we have $\tau_R(M) = \dim_K \operatorname{Hom}_R(K, M/\sum_{i=1}^d Mx_i)$. This shows in particular, that the type is base-ring independent, if M is a CM-module. More precisely, if $S = K \oplus S_1 \oplus \ldots$ is a homogeneous and finitely generated K-algebra and if R is a graded homomorphic image of S, we may consider M as a graded CM-module over R and over S and get $\tau_R(M) = \tau_S(M)$. In particular we get in this situation $\tau_R(M) = \tau_{R/xR}\,(M/xM)$ for any homogeneous M-regular element $x \in R_+$.

B) Assume now, that M is a CM-module and that R is a graded homomorphic image of a polynomial ring $S = K[\mathbf{x}_0, \ldots, \mathbf{x}_s]$. Let $0 \to F_{s-d+1} \to F_{s-d} \to \ldots \to F_0 \to M \to 0$ be a minimal graded free resolution of the S-module M. Then we have the relation $\tau_R(M) = \operatorname{rank}\,(F_{s-d+1})$. •

(2.9) Lemma. *Let K be an algebraically closed field, let $s \ge 2$ and let R be the homogeneous coordinate ring of a scheme $X \subseteq \mathbb{P}^s_K$ of $s+3$ points in semi-uniform position. Then $\tau(R) = 2$.*

Proof: We may consider R as a graded homomorphic image of the polynomial ring $S = K[\mathbf{x}_0, \ldots, \mathbf{x}_s]$ and consider a minimal graded free resolution $0 \to F_s \to F_{s-1} \to \ldots F_0 \to R \to 0$ of the graded S-module R. But then, by (2.6)B) we have $F_s \cong S(-s-2)^2$. In view of (2.8)B) this proves our claim. ■

The next result will be useful for us when we shall consider normalizations of curves.

(2.10) Lemma. *Let K be an infinite field and let $R = K \oplus R_1 \oplus R_2 \oplus \ldots \hookrightarrow \overline{R} = K \oplus \overline{R}_1 \oplus \overline{R}_2 \oplus \ldots$ be a graded birational finite integral extension of positively graded noetherian CM-domains of dimension 2. Assume that R is homogeneous and that $h^2(R)_1 = 0$. Let $\delta := h^2(R)_0 - h^2(\overline{R})_0$. Then*

a) $\dim_K(\overline{R}_n) - \dim_K(R_n) = \delta$ *for all* $n > 0$.

b) *The graded ring $\overline{R}$ is homogeneous and minimally generated as a K-algebra by $\dim_K(R_1) + \delta$ homogeneous elements of degree 1.*

Proof: We write $Q := \overline{R}/R$ and consider the short exact sequence of finitely generated and graded R-modules

$$(*) \qquad 0 \to R \to \overline{R} \to Q \to 0\ .$$

As $\overline{R}$ is a birational extension ring of R, the R-module Q is of Krull dimension ≤ 1. So, the Hilbert polynomials p_R and $p_{\overline{R}}$ of R respectively $\overline{R}$ differ only by a constant and we thus may write $p_R(n) = dn - h^2(R)_0 + 1$ and $p_{\overline{R}}(n) = dn - h^2(\overline{R})_0 + 1$ for all $n \in \mathbb{Z}$ where d denotes the degree of the scheme $\mathrm{Proj}(R)$.

By our hypothesis, $h^2(R)_n = 0$ for all $n > 0$. As $\dim(Q) < 2$, the sequence $(*)$ yields an epimorphism $H^2(R) \to H^2(\overline{R}) \to 0$ which shows that $h^2(\overline{R})_n = 0$ for all $n > 0$. As R and $\overline{R}$ are both CM, we have $H^i(R) = H^i(\overline{R}) = 0$ for $i = 0, 1$. So, for all $n \in \mathbb{N}$, we obtain $\dim_K(R_n) = p_R(n) = dn - h^2(R)_0 + 1$ and $\dim_K(\overline{R}_n) = p_{\overline{R}}(n) = dn - h^2(\overline{R})_0 + 1$. But this obviously proves claim a).

To prove statement b), let $s = \dim_k(R_1), r = \dim_K(\overline{R}_1)$ and let $\ell_1, \ldots, \ell_r$ be a K-basis of $\overline{R}_1$ such that $\ell_1, \ldots, \ell_s \in R_1$. By statement a) we know that $r = s + \delta$. It thus remains to show that $\overline{R} = K[\ell_1, \ldots, \ell_r]$. As R is homogeneous, we have $R = K[\ell_1, \ldots, \ell_s]$. It therefore is enough to show that $\overline{R} = R[\ell_{s+1}, \ldots, \ell_r]$. It suffices to prove that $\overline{R} = R + \sum_{i=s+1}^{r} R\ell_r$, hence that the graded R-module Q is generated in degree 1. As $H^0(\overline{R}) = H^1(R) = 0$, the sequence $(*)$ gives $H^0(Q) = 0$. As K is an infinite field, we thus find a Q-regular element $\ell \in R_1$. In view of statement a), the multiplication maps $\ell : Q_n \to Q_{n+1}$ are isomorphisms for all $n \in \mathbb{N}$. This proves our claim. ■

3. THE FOUR COHOMOLOGY TYPES
IN EMBEDDING DIMENSIONS ≥ 4

In this section, we give a first cohomological classification of the curves under consideration. This classification will be given in terms of the Hartshorne-Rao -module of our curve. It will turn out, that we may expect only five different types of curves to occur. In addition, one of the five types is exceptional and occurs only if the embedding dimension r equals 3.

From now on let K be an algebraically closed field, let $r \geq 3$ be an integer, and let $S = K[\mathbf{x}_0, \ldots, \mathbf{x}_r]$ be the polynomial ring in the indeterminates $\mathbf{x}_0, \ldots, \mathbf{x}_r$ over K. Finally, let $\mathcal{C} \subseteq \mathbb{P}^r_K := \mathrm{Proj}(S)$ be a non-degeneratedly embedded closed integral connected subscheme of dimension one — so we refer to $\mathcal{C}$ as a curve — and of degree $r+2$. Let $A = K \oplus A_1 \oplus \ldots$ be the homogeneous coordinate ring of $\mathcal{C}$. So A is a homogeneous, integral, two-dimensional homomorphic image of S and its first homogeneous part A_1 is a K-vector space of dimension $r+1$. As already announced in the introduction, we shall now study the structure of the Hartshorne-Rao module $H^1(A)$ of our curve $\mathcal{C}$.

(3.1) Remark. In view of the Serre-Grothendieck conrrespondence (cf. (2.2)) we have $H^1(\mathcal{C}, \mathcal{O}_c) \cong H^2(A)_0$, so that the arithmetic genus of $\mathcal{C}$ is given by $h^2(A)_0$. Consequently the Hilbert polynomial of $\mathcal{C}$ takes the form $\chi_{\mathcal{C}}(t) = (r+2)t + 1 - h^2(A)_0$. •

(3.2) Remark. Let $\ell \in A_1 \setminus \{0\}$. As A is a domain of dimension 2, we have $H^0(A) = 0$ and $\dim(A/\ell A) = 1$, hence $H^2(A/\ell A) = 0$. So, if we apply local cohomology to the short exact sequence

$$0 \to A \xrightarrow{\ell} A(1) \to A/\ell A(1) \to 0$$

we obtain for each $n \in \mathbb{Z}$ an exact sequence

$$\begin{aligned} 0 \to H^0(A/\ell A)_{n+1} \to H^1(A)_n \xrightarrow{\ell} H^1(A)_{n+1} \\ H^1(A/\ell A)_{n+1} \to H^2(A)_n \xrightarrow{\ell} H^2(A)_{n+1} \to 0 \ . \end{aligned}$$ •

(3.3) Lemma. *Let $\ell \in A_1 \setminus \{0\}$. Then:*
- a) $h^1(A/\ell A)_n = r+2$ *for all* $n < 0$.
- b) $h^1(A/\ell A)_0 = r+1$.
- c) $h^1(A/\ell A)_1 = 2$.
- d) $h^1(A/\ell A)_2 \leq 1$.
- d′) $h^1(A/\ell A)_2 = 0$ *if the given element ℓ is a generic linear form.*
- e) $h^1(A/\ell A)_n = 0$ *for all* $n > 2$.

Proof: Let $B := (A/\ell A)/H^0(A/\ell A)$ and let $\mathbb{P}^{r-1}_K \subseteq \mathbb{P}^r_K$ be the hyperplane defined by ℓ. Then, $X := \mathrm{Proj}(B) = \mathcal{C} \cap \mathbb{P}^{r-1}_K \subseteq \mathbb{P}^{r-1}_K$ is a zero-dimensional subscheme of degree $r+2$, and therefore $\mathcal{O}_X$ has length $r+2$. So, if we apply (2.3)a) with $R = A$ and $M = B$, we see that $d(B)_n = r+2$ for all $n \in \mathbb{Z}$. Let b_n denote the dimension of the K-vector space B_n. Then, the graded short exact sequence $0 \to B \to D(B) \to H^1(B) \to 0$ together with the natural homogeneous isomorphism $H^1(A/\ell A) \xrightarrow{\cong} H^1(B)$ shows that $h^1(A/\ell A)_n = r+2-b_n$ for all $n \in \mathbb{Z}$. As $b_n = 0$ for all $n < 0$ and as $b_0 = 1$, this proves statements a) and b).

By the Serre-Grothendieck correspondence (2.2) we have $D(A)_0 = \Gamma(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = K = A_0$ and so the natural short exact sequence $0 \to A \to D(A) \to H^1(A) \to 0$ yields $H^1(A)_0 = 0$. If we apply the exact sequence of (3.2) with $n = 0$, we thus get that $H^0(A/\ell A)_1 = 0$. Consequently we have isomorphisms $B_1 \cong (A/\ell A)_1 \cong$

$A_1/\ell A_0 \cong K^{r+1}/K \cong K^r$, which show that $b_1 = r$ hence that $h^1(A/\ell A)_1 = r+2-r = 2$. This proves statement c).

If now, we apply (2.3)b) with $R = A,\ M = B$ and $n = 1$, we immediately obtain statement d). Applying the same result for all $n >$

(3.6) Lemma. *Let $r \geq 4$ and $h^1(A)_1 \neq 0$. Then $h^1(A)_2 < h^1(A)_1$.*

Proof: Let $\ell \in A_1 \setminus \{0\}$ be a generic linear form. Then (3.2) and (3.3)d$'$) give rise to a short exact sequence $0 \to H^0(A/\ell A)_2 \to H^1(A)_1 \xrightarrow{\ell} H^1(A)_2 \to 0$. Assume that $h^1(A)_2 \geq h^1(A)_1$. Then this sequence shows that $H^0(A/\ell A)_2 = 0$. By (3.5)b) we thus get $H^0(A/\ell A) = 0$. So (3.2) leads to an exact sequence $0 \to H^1(A)_2 \to H^1(A)_3 \to H^1(A/\ell A)_3$. In view of (3.3)e) we therefore have $h^1(A)_3 = h^1(A)_2 = h^1(A)_1 \neq 0$ and hence a contradiction to (3.4)d). ■

(3.7) Lemma. *Let $r \geq 4$ and let $h^1(A)_2 \neq 0$. Then:*

a) $h^1(A)_2 = 1, h^1(A)_1 = 2$.

b) $h^2(A)_0 = 0$.

Proof: Obvious from (3.4)b), (3.4)c) and (3.6). ■

If M is a graded A-module, we denote its (graded) *socle* by $\mathrm{soc}(M)$, so that $\mathrm{soc}(M)$ is the sum of all graded simple submodules of M and may be written as $0 :_M A_+$. Using this notation, we have.

(3.8) Lemma. *Let $r \geq 4$ and let $h^1(A)_2 \neq 0$. Then $soc(H^1(A)) = H^1(A)_2$.*

By (3.4)d) we obviously have $H^1(A)_2 \subseteq \mathrm{soc}(H^1(A))$. Assume that $H^1(A)_2 \neq \mathrm{soc}(H^1(A))$. Then (3.4)a) shows that $H^1(A)_1 \cap \mathrm{soc}(H^1(A)) =: T \neq 0$. Now, let $\ell \in A_1 \setminus \{0\}$ be a generic linear form. Then (3.2) and (3.3)d$'$) give rise to a short exact sequence $0 \to H^0(A/\ell A)_2 \to H^1(A)_1 \xrightarrow{\ell} H^1(A)_2 \to 0$. So, by statement (3.7)a) we first obtain that $h^0(A/\ell A)_2 = 1$, and from this we conclude by (3.5)b) that $H^0(A/\ell A)$ is a cyclic A-module. Moreover we have $0 \neq T \subseteq 0 :_{H^1(A)_1} \ell \cong H^0(A/\ell A)_2$. As $h^0(A/\ell A)_2 = 1$ we get $T = 0 :_{H^1(A)_1} \ell$, and so the connecting homomorphism $\delta : H^0(A/\ell A)(1) \to H^1(A)$ maps $H^0(A/\ell A)_2$ onto T. By (3.2) and (3.4)d) we see that δ maps $H^0(A/\ell A)_3$ onto $H^1(A)_2$. Again by (3.4)d), this yields a graded epimorphism $H^0(A/\ell A) \to T \oplus H^1(A)_2 = \mathrm{soc}(H^1(A))$, which shows that $\mathrm{soc}(H^1(A))$ is a cyclic A-module and sits in more than one degree. This is obviously a contradiction. ■

Now, as $h^1(A)_n = 0$ for all $n \neq 1, 2$ it seems natural to distinguish different cases according to the values taken by $h^1(A)_1$ and $h^1(A)_2$. We therefore call the pair $(h^1(A)_1, h^1(A)_2) \in \mathbb{N}_0^2$ the *numerical cohomology type* (n.c.t) of A. Now we have:

(3.9) Theorem. *Let $r \geq 4$. Then, only the four numerical cohomology types $(0,0), (1,0), (2,0)$ and $(2,1)$ may be expected to occur. Moreover according to these four cases, we have the following statements on the structure of the ring A, the structure of the A-module $H^1(A)$ and on the arithmetic genus $h^2(A)_0$ of the curve $\mathcal{C}$.*

case	*n.c.t*	*structure of* $H^1(A)$	*structure of* A	$p_a(C) = h^2(A)_0$
I	$(0,0)$	$H^1(A) = 0$	A *is a CM-ring of type* 2	2
II	$(1,0)$	$H^1(A) \cong K(-1)$	A *is a Buchsbaum ring with invariant* 1	1
III	$(2,0)$	$H^1(A) \cong K^2(-1)$	A *is a Buchsbaum ring with invariant* 2	0
IV	$(2,1)$	$soc(H^1(A)) = H^1(A)_2 = K(-2)$	A *is a 2-Buchsbaum ring*	0

Proof: From (3.4)b) and c) it follows that the only numerical cohomology types which may be expected besides the four listed above are $(1,1)$ and $(2,2)$. But these two possibilities are excluded by (3.6). Now, the statements on the structure of $H^1(A)$ follow by (3.4)a) and d) (in the cases I, II, III) and by (3.8) (in the case IV). As A is a homogeneous domain of dimension 2, the statements on the structure of A are an easy consequence of the corresponding statements on the structure of $H^1(A)$ – at least in the cases II, III, IV. In the case I it also follows at once, that A is a CM-ring. As the statements on $h^2(A)_0$ are obvious from (3.4) it remains to be shown that the type $\tau(A)$ of A equals 2, if A is a CM-ring.

To do so, let $\ell \in A_1 \setminus \{0\}$ be a generic linear form, so that $X := \mathrm{Proj}(A/\ell A)$ is a scheme of $r+2$ points in semi-uniform position in the hyperplane $\mathbb{P}^{r-1}_K \subseteq \mathbb{P}^r_K = \mathrm{Proj}(S)$ defined by ℓ. As $A/\ell A$ is a CM-ring, it is the homogeneous coordinate ring of X. So (2.9), applied with $s = r-1$ and $R = A/\ell A$ shows that $\tau(A/\ell A) = 2$. In view of (2.8)A) we thus get $\tau(A) = 2$. ■

(3.10) Remark. Let $r \geq 4$. The data given in the table of Theorem (3.9) define indeed the structure of $H^1(A)$ as a module over A, e.g. as a module over $S := K[\mathbf{x}_0, \ldots, \mathbf{x}_r]$. Obviously we only have to make this evident in the case IV. As the K-vector spaces $H^1(A)_1$ and $H^1(A)_2$ are of dimension 2 and 1 respectively and as $H^1(A)_1 \setminus \{0\}$ contains no socle element of $H^1(A)_1$, the K-vector space

$$V := \left\{\ell \in S_1 \mid \ell H^1(A)_1 = 0\right\} = 0 :_{S_1} H^1(A)$$

is of dimension $r-1$. After an appropriate change of coordinates we may write $V = \sum_{j=2}^r K\mathbf{x}_j$ and consider $H^1(A)$ as a module over the polynomial ring $T = K[\mathbf{x}_0, \mathbf{x}_1]$. Now, the kernels of the two multiplication maps $\mathbf{x}_j : H^1(A)_1 \to H^1(A)_2$, $(j = 0,1)$ are both of dimension one and different from each other.

As an easy consequence we see that $H^1(A)$ has a minimal graded free T-resolution of the shape

$$0 \to T(-4) \xrightarrow{\nu} T(-2)^3 \xrightarrow{\mu} T(-1)^2 \to H^1(A) \to 0$$

in which the maps ν and μ are given respectively by the matrices

$$[-\mathbf{x}_1^2, \mathbf{x}_0^2, \mathbf{x}_1\mathbf{x}_0], \quad \begin{bmatrix} \mathbf{x}_0, & 0, & \mathbf{x}_1 \\ 0, & \mathbf{x}_1, & \mathbf{x}_0 \end{bmatrix} .$$

Another consequence is, that we have an isomorphism of graded T-modules

$$H^1(A) \cong \mathrm{Hom}_K\big(T/(\mathbf{x}_0, \mathbf{x}_1)^2 T, K\big)\ (-2)\ .$$

Observe is particular, that up to graded isomorphism, $H^1(A)$ is uniquely determined by $V = 0 :_{S_1} H^1(A)$. •

(3.11) Remark. A) Besides the previous proposition, the results of this section give rise to a few observations, which also hold true if $r = 3$. As already remarked in the proof of (3.9) we cannot exclude the two additional numerical cohomology types $(1,1)$ and $(2,2)$ if $r = 3$. By (3.3)d′) and (3.2) we see that $\mathrm{soc}(H^1(A)) = H^1(A)_2$ in these two cases. In particular A is a 2-Buchsbaum ring in these two cases. In case of the numerical cohomology type $(2,1)$ we see again by (3.3)d′) and (3.2) that $\mathrm{soc}(H^1(A))_1 \leq 1$. But for $r = 3$ we cannot exclude the case $\mathrm{soc}(H^1(A))_1 = 1$. (Here $\mathrm{soc}(\mathcal{U})_n$ denotes the K-dimension of the n-th homogeneous part of the socle of a graded A-module $\mathcal{U}$).

B) In case of the numerical cohomology type $(0,0)$, it is always true, that A is a CM-ring of type 2, simply as Lemma (2.9), which we used to calculate $\tau(A)$, holds also true for $s = 2$. So, we may say in general, that $\mathcal{C}$ is arithmetically CM if and only if $p_a(\mathcal{C}) = 2$ and that in this case we must have $\tau(A) = 2$.

C) If $\mathcal{C}$ is smooth, it is of numerical cohomology type (0,0) precisely if it is of (geometric) genus 2. The curves of this class present what one could call the "generic case" in our classification.

D) Altogether, let us notice, that besides the 4 cases listed in (3.9), we cannot exclude at the moment the following 3 additional cases if $r = 3$:

case	*n.c.t*	*soc* $(H^1(A))$	*structure of A*	$p_a(C) = h^2(A)_0$
II′	(1,1)	$K(-2)$	*A is a 2-Buchsbaum ring*	1
IV′	(2,1)	$K(-1) \oplus K(-2)$	*A is a 2-Buchsbaum ring*	0
IV″	(2,2)	$K(-2)^2$	*A is a 2-Buchsbaum ring*	0

E) Later we shall use the structure of the minimal free resolution of the Hartshorne-Rao module $H^1(A)$ to exclude the cases II′ and IV′ (s.(5.6)). On the other hand, the case IV″ occurs indeed, as shown by the example $\mathcal{C} = \mathrm{Proj}\big(K[s^5, s^4t, st^4, t^5]\big)$ in which s and t are used to denote indeterminates (s.[S₂]). •

4. Embeddings and projections of smooth curves

In this section, we describe the previous cohomologically defined types of curves in geometric terms: By means of embeddings or projections of smooth curves of simple nature.

We keep the previous hypotheses and notations so that K is an algebraically closed field, S denotes the polynomial ring $K[\mathbf{x}_0, \ldots, \mathbf{x}_r]$, $\mathcal{C} \subseteq \mathbb{P}^r_K = \operatorname{Proj}(S)$ denotes a curve (e.g. an integral closed connected nondegenerate subscheme of dimension 1) of degree $r+2$ with homogeneous coordinate ring A. Moreover we set $r' := r + h^1(A)_1$. By (3.4)b) we have $r \leq r' \leq r+2$.

(4.1) Remark and Definition. A) In view of (3.5)a), the canonical exact sequence $0 \to A \xrightarrow{\eta} D(A) \to H^1(A) \to 0$ shows that the graded A-module $D(A)$ is generated by homogeneous elements of degree one and that $d(A)_1 = r'+1$. In particular $D(A)$ is a homogeneous integral K-algebra of dimension 2 and minimally generated by $r'+1$ homogeneous elements of degree one. Therefore $D(A)$ may be viewed as a graded homorphic image of the polynomial ring $S' := K[\mathbf{x}_0, \ldots, \mathbf{x}_{r'}]$ and so $\mathcal{C}' = \operatorname{Proj}(D(A))$ naturally occurs as a nondegeneratedly embedded curve in $\mathbb{P}^{r'}_K = \operatorname{Proj}(S')$. We call this curve the curve which is *associated to* $\mathcal{C}$.

B) As $H^1(A)$ vanishes in large degrees, the curves $\mathcal{C}'$ and $\mathcal{C}$ have both the same degree $r+2$ and the canonical homomorphism $\eta : A \to D(A)$ induces an isomorphism $\varepsilon : \mathcal{C}' \to \mathcal{C}$. If $h^1(A)_1 = r' - r > 0$ we write P for the linear subspace $\mathbb{P}^{r'-r-1}_K = \operatorname{Proj}\big(S'/(\mathbf{x}_0, \ldots, \mathbf{x}_r)\big)$ of $\mathbb{P}^{r'}_K$. If $h^1(A)_1 = 0$, we set $P = \emptyset$. So P is empty, a point or a line respectively if $r' = r, r' = r+1$ or $r' = r+2$. Moreover we have $P \cap \mathcal{C}' = \emptyset$. If $\pi_P : \mathbb{P}^{r'}_K \setminus P \to \mathbb{P}^r$ denotes the projection from the center P $\big($ given on closed points by $(c_0 : \ldots : c_{r'}) \mapsto (c_0 : \ldots : c_r)\big)$, the isomorphism ε is induced by the projection π_P. Let $\operatorname{Sec}(\mathcal{C}') \subseteq \mathbb{P}^{r'}_{\mathrm{K}}$ be the secant variety of $\mathcal{C}'$. Then, as ε is an isomorphism, we have $P \cap \operatorname{Sec}(\mathcal{C}') = \emptyset$. •

In the sequel, we use the notion of $*$canonical module of $D(A)$, as it is defined in [Bru-He, (3.6.8)]. In our context, the $*$canonical module of $D(A)$ is unique up to an isomorphism of graded modules (s.[Bru-He, (3.6.9)]). So, we allow ourselves to speak of "the" $*$canonical module of $D(A)$.

(4.2) Proposition. *$D(A)$ is CM-ring. Moreover, for all $r \geq 3$ we have:*

a) *If $h^1(A)_1 = 0$, then $A = D(A)$ so that $D(A)$ is of type 2. In this case, the $*$canonical module of $D(A)$ is generated by (two) homogeneous elements of degree 0.*

b) *If $h^1(A)_1 = 1$, then $D(A)$ is a Gorenstein ring and its $*$canonical module is generated by a homogeneous element of degree 0, hence equal to $D(A)$.*

c) *If $h^1(A)_1 = 2$, after an $\mathcal{C}' \subseteq \mathbb{P}^{r+2}_K$ is a normal rational curve of degree $r+2$ so that $D(A)$ is of type $r+1$ with a $*$canonical module which is generated by $(r+1)$ homogeneous elements of degree 1.*

Proof: As $H^0(D(A)) = H^1(D(A)) = 0$ and $\dim(D(A)) = 2, D(A)$ is a CM-ring.

a) If $h^1(A)_1 = 0$ we have $H^1(A) = 0$ so that $D(A) = A$. Now, by (3.9) and (3.11)B) we see that $\tau(D(A)) = 2$. Next, let $s \in \mathbb{N}$, let $A^{(s)}$ be the t-th Veronese subring of A and let $\mathcal{C}^{(s)} = \operatorname{Proj}(A^{(s)})$. Then, the Hilbert-polynomial of $\mathcal{C}^{(s)}$ has the shape $\chi_{\mathcal{C}^{(s)}}(t) = \chi_{\mathcal{C}}(st) = st(r+2) - 1$ (s.(3.1), (3.4)b)). Moreover we have $H^1(A^{(s)}) = H^1(A)^{(s)} = 0$. In particular we see that $\big(\dim_K A^{(s)}\big)_1 = \chi_{\mathcal{C}^{(s)}}(1) = s(r+2) - 1$, so that $\mathcal{C}^{(s)}$ is of embedding dimension $s(r+2) - 2$ and of degree $s(r+2)$. So, we may apply what we know already to $A^{(s)}$ and get that $\tau(A^{(s)}) = 2 = \tau(A)$. By [Bru-He, (3.3.11)] this means that the $*$canonical module $\Omega^{(s)}$ of $A^{(s)}$ is minimally generated by the same number of homogeneous elements as the $*$canonical module Ω of A for all $s \in \mathbb{N}$. By [Bru-He, (3.6.21)(c)], we have $\Omega^{(s)} = \oplus_{n \in \mathbb{Z}} \Omega_{ns}$. As Ω is a CM-module, $\Omega^{(s)}$ therefore is only minimally generated by two homogeneous elements for all $s \in \mathbb{N}$ if Ω is generated in degree 0.

b) Let us first recall, that a non-degenerate arithmetically CM curve of degree $t+1$ in $\mathbb{P}^t_K$ is arithmetically Gorenstein (see for example [Ho-St-V, Theorem B, (iv)]. As $h^1(A)_1 = 1$, we have $r' = r+1$, so that $\mathcal{C}'$ is an arithmetically CM-curve of degree $r'+1$ in $\mathbb{P}_K^{r'}$ and hence is arithmetically Gorenstein, So, $D = D(A)$ is a Gorenstein ring. Now, let $s \in \mathbb{N}$ and consider again the Veronese transform $\mathcal{C}'^{(s)} = \mathrm{Proj}(D^{(s)})$, whose Hilbert polynomial is given by $\chi_{\mathcal{C}'^{(s)}}(t) = \chi_{\mathcal{C}'}(st) = \chi_{\mathcal{C}}(st) = st(r+2)$ (s. (3.1), (3.4)b)). Now, we conclude again, that $\mathcal{C}'^{(s)}$ is an arithmetically CM-curve of degree $s(r+2)$ and of embedding dimension $s(r+2)-1$, hence is arithmetically Gorenstein. As this holds for all $s \in \mathbb{N}$, we conclude as in the proof of statement a) that D has $*$canonical module generated in degree 0.

c) Let $p_1, \ldots, p_{r+2} \in \mathcal{C}'$ be $r+2$ distinct closed points. Then, there is a hyperplane $H \subseteq \mathbb{P}_K^{r+2}$ which contains all these points. By Bezout we conclude that the intersection multiplicity of $\mathcal{C}'$ with H in each of these points p_i equals one. So, all these points are regular points of $\mathcal{C}'$, hence $\mathcal{C}'$ must be smooth. By (3.4)b) we see that the arithmetic genus $p_a(\mathcal{C}') = h^2(D(A))_0 = h^2(A)_0$ equals 0, so that $\mathcal{C}'$ is smooth and rational. Therefore, there is some closed embedding $\iota : \mathbb{P}_K^1 \hookrightarrow \mathbb{P}_K^{r+2}$, such that $\mathcal{C}' = \iota(\mathbb{P}_K^1)$. This emebdding is given by a divisor of some degree $r'' > 0$ on $\mathbb{P}_K^1$. If we write $\mathbb{P}_K^1 = \mathrm{Proj}(K[s,t])$ with two indeterminates s and t, we thus get a homomorphism $K[s,t]^{(r'')} \xrightarrow{\alpha} D(A)$ of graded rings, which is an isomorphism in large degrees. As $D(A)$ is a CM-ring α must be an isomorphism and we must have $r'' = r+2$. Now, our claim is obvious in view of [Br-He, (3.6.21)(c), (3.6.10)]. ■

Now, we want to give a first geometric description of $\mathcal{C}$ according to the four cases I, II, III, IV of (3.9) and the exceptional case IV$''$ of (3.11) C).

(4.4) Remark. For a point p of a noetherian scheme X we write $m_p(X)$ for the multiplicity of X at p. Then, the same argument as made in the beginning of the proof of statement (4.2)c) tells us, that $\sum_{p' \in \mathcal{C}'}(m_{p'}(\mathcal{C}') - 1) \leq r + 2 - r' = 2 - h^1(A)_1 = h^2(A)_0$. As $\varepsilon : \mathcal{C}' \to \mathcal{C}$ is an isomorphism, we thus can say:

(i) If $h^1(A)_1 = 0$, then $\mathcal{C}$ is of arithmetic genus 2 and has at most 2 singularities; in case of two singularities, both must be double points, in the case of one singularity p, we must have $m_p(\mathcal{C}) \leq 3$.

(ii) If $h^1(A)_1 = 1$, then $\mathcal{C}$ is of arithmetic genus 1 and has at most one singularity and this singularity must be a double point.

(iii) If $h^1(A)_1$

homogeneous ring with $\dim_K\big(D(A)_1\big) = d(A)_1 = r' + 1 = r + h^1(A)_1 + 1$. As $h^2(A)_0 = h^2\big(D(A)\big)_0$ we now can apply (2.10) to the rings $D(A) \subseteq B$ in order to see:

> B is a homogeneous ring which is minimally generated as a K-algebra by
> $r' + \delta(\mathcal{C}) + 1$ homogeneous elements of degree 1.

E) Let $\tilde{r} = r' + \delta(\mathcal{C})$ so that $r' \leq \tilde{r} \leq r + 2$. Then, by the last observation there is a non- degenerate closed immersion˜

(4.6) Proposition. *Let $r \geq 3$. Then*

a) *In the case I_1, $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+1}_K$ is an elliptic normal curve and $\mathcal{C}$ is obtained by projecting $\tilde{\mathcal{C}}$ from a point $\mathbb{P}^0_K \in \mathbb{P}^{r+1}_K$ lying on presicely one secant line to $\tilde{\mathcal{C}}$; moreover this secant line is simple.*

b) *In the case I_2, $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+2}_K$ is a rational normal curve and $\mathcal{C}$ is obtained by projecting $\tilde{\mathcal{C}}$ from a line $\mathbb{P}^1_K \subseteq \mathbb{P}^{r+2}_K$ lying on precisely one trisecant plane to $\tilde{\mathcal{C}}$. The unique singularity p of $\mathcal{C}$ satisfies $m_p(\mathcal{C}) = 3$.*

c) *In the case I'_2, $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+2}_K$ is as an statement b) and $\mathcal{C}$ is obtained by projecting $\tilde{\mathcal{C}}$ from a line which is intersected by precisely two lines secant to $\tilde{\mathcal{C}}$; moreover these two secant lines are simple and skew to each other.*

d) *In the case II_1, $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+2}_K$ is again as in statement b) and $\mathcal{C}$ is obtained by projecting $\tilde{\mathcal{C}}$ from a line $\mathbb{P}^1_K \subseteq \mathbb{P}^{r+2}_K$ which is intersected by precisely one secant line to $\tilde{\mathcal{C}}$; moreover this secant line is simple.*

Proof. a) By statement E) of (4.5) we may indeed consider $\tilde{\mathcal{C}}$ as a non-degenerate arithmetically normal curve of degree $r+2$ in $\mathbb{P}^{r+1}_K$. In the notation of (4.5) the genus of $\tilde{\mathcal{C}}$ is given by $h^2(B)_0 = h^2(A)_0 - \delta(\mathcal{C}) = 1$ so that $\tilde{\mathcal{C}}$ is an elliptic normal curve. By (4.5)E) and as $\mathcal{C} = \mathcal{C}'$ (s.(4.3)a)) the morphism $\nu : \tilde{\mathcal{C}} \to \mathcal{C}$ is induced by a linear projection $\mathbb{P}^{r+1}_K \backslash \mathbb{P}^0_K \to \mathbb{P}^r_K$ from an appropriate point $\mathbb{P}^0_K \in \mathbb{P}^{r+1}_K \backslash \tilde{\mathcal{C}}$. As $\mathcal{C}$ has a unique singularity, which moreover is a double point (s.(4.5)F)), the center of projection $\mathbb{P}^0_K$ must ly on precisely one secant line to $\tilde{\mathcal{C}}$, which moreover must be bisecant.

b), c) By (4.5)E) we may consider $\tilde{\mathcal{C}}$ as a non-degenerate arithmetically normal curve of degree $r+2$ in $\mathbb{P}^{r+2}_K$ with genus $h^2(B)_0 = h^2(A)_0 - \delta(\mathcal{C}) = 0$, hence a rational normal curve. Again by (4.5)E) and as $\mathcal{C} = \mathcal{C}'$, the morphism $\nu : \tilde{\mathcal{C}} \to \mathcal{C}$ is induced by a projection $\lambda : \mathbb{P}^{r+2}_K \backslash \mathbb{P}^1_K \to \mathbb{P}^r_K$ from a line $L := \mathbb{P}^1_K \subseteq \mathbb{P}^{r+2}_K$ disjoint to $\tilde{\mathcal{C}}$. For each $k \in \mathbb{N}$, the assignement $\mathbb{P}^2_K \mapsto \lambda(\mathbb{P}^2_K \backslash L)$ gives rise to an injective map from the set of all k-secant planes $\mathbb{P}^2_K \supseteq L$ of $\tilde{\mathcal{C}}$ into the set of all k-fold points of $\lambda(\tilde{\mathcal{C}}) = \mathcal{C}$.

In the case $I_{2'}$, we may use the dates listed in (4.5)F) to conclude that L contains precisely two 2-secant planes of $\tilde{\mathcal{C}}$ and no k-secant plane of $\tilde{\mathcal{C}}$ with $k > 2$. This immediately proves statement c).

Assume now, that we are in case I_2. Then, by (4.5)F) we see that the unique singularity p of $\mathcal{C}$ satisfies $m_p(\mathcal{C}) = 2$ or $m_p(\mathcal{C}) = 3$. If $m_p(\mathcal{C}) = 2$, the above injection shows that L contains precisely one 2-secant plane $H := \mathbb{P}^2_K$ of $\tilde{\mathcal{C}}$ and no k-secant plane of $\tilde{\mathcal{C}}$ with $k > 2$. This means, that L is intersected in some point Q by a 2-secant line L' of $\mathcal{C}$. Now, let $\varrho : \mathbb{P}^{r+2}_K \backslash Q \to \mathbb{P}^{r+1}_K$ be the projection from Q, let $S := \varrho(L \backslash Q)$ and let $\sigma : \mathbb{P}^{r+1}_K \backslash S \to \mathbb{P}^r_K$ be the projection from S. Then $\lambda = \sigma \circ \big(\varrho \upharpoonright_{\mathbb{P}^{r+2}_K \backslash L}\big)$ and $\varrho(\tilde{\mathcal{C}}) \subseteq \mathbb{P}^{r+1}_K$ is a nondegenerate curve with $q := \varrho(L' \backslash Q)$ as a unique singularity and moreover $m_q\big(\varrho(\mathcal{C})\big) = 2$. As $\sigma\big(\varrho(\tilde{\mathcal{C}})\big) = \mathcal{C}$ has p as a unique singularity we obtain $\sigma(q) = p$. As $m_p(\mathcal{C}) = 2$ we conclude that the line through S and q intersects $\varrho(\tilde{\mathcal{C}})$ only in q and the corresponding multiplicity of intersection is $2 = m_q\big(\varrho(\tilde{\mathcal{C}})\big)$. It follows that $\sigma(q') \neq p$ hence that $\sigma(q') \in \mathcal{C}$ is non-sigunlar for each $q' \in \varrho(\tilde{\mathcal{C}}) \backslash \{q\}$. So, for each such q', the line through S and q' intersects $\varrho(\tilde{\mathcal{C}})$ only in q' and this intersection is transversal. Therefore the projection σ induces an isomorphism $\sigma \upharpoonright : \varrho(\tilde{\mathcal{C}}) \xrightarrow{\cong} \mathcal{C}$. This shows that $\mathcal{C} \subseteq \mathbb{P}^r_K$ is not linearly normal and hence contradicts $H^1(A)_1 = 0$.

So, we must have $m_p(\mathcal{C}) = 3$. In view of the previously observed relation between k-secant planes $\mathbb{P}^2_K \supseteq L$ to $\tilde{\mathcal{C}}$ and k-fold points of $\mathcal{C}$, statement b) follows immediately.

d) By (4.5)C) and (4.5)E) we may conclude as in the proof of statements b) and c), that $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+2}_K$ is a rational normal curve. Moreover the morphism $\eta : \tilde{\mathcal{C}} \to \mathcal{C}'$ is induced by a projection $\lambda : \mathbb{P}^{r+2}_K \backslash Q \longrightarrow \mathbb{P}^{r+1}_K$ from a point $Q \in \mathbb{P}^{r+2}_K \backslash \tilde{\mathcal{C}}$. As $\mathcal{C}'$ has precisely one singularity, which moreover is a double point (s.(4.4)), Q is contained in precisely one secant line H; moreover, H is a simple secant line to $\mathcal{C}$. In addition, we know by (4.3)b) that the isomorphism $\varepsilon : \mathcal{C}' \to \mathcal{C}$ is induced by the projection $\mathbb{P}^{r+1}_K \backslash P \to \mathbb{P}^r_K$ from a point $P \in \mathbb{P}^{r+1}_K \backslash \mathrm{Sec}(\mathcal{C}')$. Now, let $L = \mathbb{P}^1_K \subseteq \mathbb{P}^{r+2}_K$ be the unique line passing through Q and satisfying $\lambda(L \backslash Q) = P$. Then, $\nu = \varepsilon \circ \eta$ is induced by a projection $\mathbb{P}^{r+2} \backslash L \to \mathbb{P}^r_K$ from L, and H is the only secant of $\tilde{\mathcal{C}}$ which interersects L. This proves statement d). ■

(4.7) Remark. A) Keep the previous notations and hypothesis. Then, we see from (4.3) and (4.6), that exept in the case I_0, the curve $\mathcal{C}$ always is a projection of a rational normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{\tilde{r}}_K$ or an elliptic normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{\tilde{r}}_K$. More precisely:

(i) In the cases I_1 and II_0, the curve $\mathcal{C}$ is obtained by projecting an elliptic normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+1}_K$ from a point $P \in \mathbb{P}^{r+1}_K$. The case II_0 is characterized by the condition $P \notin \mathrm{Sec}(\tilde{\mathcal{C}})$.

(ii) In the cases I_2, I'_2, II_1, III, IV and IV$''$, the curve $\mathcal{C}$ is obtained by projecting a rational normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+2}_K$ from a line $L \subseteq \mathbb{P}^{r+2}_K$. The distinction between the 6 possible cases is caused by the different posititions of the projecting line L with respect to the secants of $\tilde{\mathcal{C}}$.

Observe that in the case IV$''$ the condition that L is contained in a quadrisecants 3-space $\mathbb{P}^3_K \subseteq \mathbb{P}^5_K$ to $\tilde{\mathcal{C}}$ may be omitted, as it is always satisfied.

B) For the sake of completeness let us mention here how non-degenerate projective curves $\overline{\mathcal{C}} \subseteq \mathbb{P}^r_K$ of degree $r+1$ may be described in geometric terms. Then, three possible cases $\overline{\mathrm{I}}, \overline{\mathrm{II}}$ and $\overline{\mathrm{III}}$ may occur:

$\overline{\mathrm{I}}$: $\overline{\mathcal{C}}$ is an elliptic normal curve.

$\overline{\mathrm{II}}$: $\overline{\mathcal{C}}$ is obtained by projecting a rational normal curve $\tilde{\overline{\mathcal{C}}} \subseteq \mathbb{P}^{r+1}_K$ from a point $P \in \mathbb{P}^{r+1}_K \backslash \mathrm{Sec}(\tilde{\overline{\mathcal{C}}})$.

$\overline{\mathrm{III}}$: $\overline{\mathcal{C}}$ is obtained by projecting a rational normal curve from a point $P \in \mathbb{P}^{r+1}_K \backslash \tilde{\overline{\mathcal{C}}}$ which lies on precisely one secant line L to $\tilde{\overline{\mathcal{C}}}$; moreover, L is a simple secant line: in addition $\overline{\mathcal{C}}$ has precisely one singular point $\overline{p}$ and this point $\overline{p}$ is a double point.

At least if $r \geq 4$, this may be seen by the arguments which were used to prove (4.3) and (4.6), as then $\overline{\mathcal{C}}$ is the associated curve of a curve $\overset{\circ}{\mathcal{C}} \subseteq \mathbb{P}^{r-1}_{\overline{K}}$ of degree $r+1$. Another approach consists in applying [Ho-St-V], (2.10) and (4.5) and proving our claim independently in the spirit of the arguments used in the proofs of (4.3) and (4.6). •

5. BETTI NUMBERS OF $D(A)$

Now, we begin with the study of the homological aspect of our curves. We keep our previsou notation. Our goal is to determine or at least to appropriate the Betti numbers of the homogeneous coordinate ring A of $\mathcal{C}$. The present section shall pave the way to this goal by an investigation of the Betti numbers of the S-module $D(A)$.

By (4.1)A) we know, that $D(A)$ is the homogeneous coordinate ring of the associated curve $\mathcal{C}' \subseteq \mathbb{P}_K^{r'}$ and hence may be written as a graded homomorphic image of the polynomial ring $S' := K[\mathbf{x}_0, \ldots, \mathbf{x}_{r'}]$, where $r' = r + h^1(A)_1$.

We first compute the graded Betti modules $\operatorname{Tor}_i^{S'}(K, D(A))$ of the S'-module $D(A)$.

(5.1) Proposition. *Let $r \geq 3$. Then:*

a) *If $h^1(A)_1 = 0$ (e.g. in the case I), $r' = r$, $S' = S$, $D(A) = A$ and*

$$\operatorname{Tor}_i^{S'}(K, D(A)) \cong \begin{cases} K & \text{for } i = 0 \\ K^{c_i}(-i-1) & \text{for } 1 \leq i \leq r'-3 \ , \\ K^{r'-1}(-r') \oplus K^{r'-2}(-r'+1) & \text{for } i = r'-2 \ , \\ K^2(-r'-1) & \text{for } i = r'-1 \ , \end{cases}$$

where $c_i = \binom{r'}{i+1} - 2\binom{r'-1}{i-1}$ *for* $1 \leq i \leq r'-3$.

b) *If $h^1(A)_1 = 1$ (e.g. in the cases II and II′)*

$$\operatorname{Tor}_i^{S'}(K, D(A)) \cong \begin{cases} K & \text{for } i = 0 \ , \\ K^{c_i}(-i-1) & \text{for } 1 \leq i \leq r'-2 \ , \\ K(-r'-1) & \text{for } i = r'-1 \ , \end{cases}$$

where

$$c_i = (r'+1)\binom{r'-1}{i} - \binom{r'+1}{i+1} \quad \text{for} \quad 1 \leq i \leq r'-2 \ .$$

c) *If $h^1(A)_2 = 2$, (e.g. in the cases III, IV, IV′ and IV″),*

$$\operatorname{Tor}_i^{S'}(K, D(A)) \cong \begin{cases} K & \text{for } i = 0 \\ K^{c_i}(-i-1) & \text{for } 1 \leq i \leq r'-1 \ , \end{cases}$$

where

$$c_i = i\binom{r'}{i+1} \quad \text{for} \quad 1 \leq i \leq r'-1 \ .$$

Proof. a) Clearly, in this case $r = r', S' = S$ and $A = D(A)$. Let $\ell \in S_1$ denote a generic linear form. As ℓ acts as a regular element on S' and on $D(A)$ we have

$$\operatorname{Tor}_i^{S'/\ell S'}(K, D(A)/\ell D(A)) \cong \operatorname{Tor}_i^{S'}(K, D(A)) \quad \text{for all} \quad i \ .$$

As $D(A)$ is CM (s.(4.2)), $D(A)/\ell D(A)$ is the homogeneous coordinate ring of $r'+2$ points in semi-uniform position in $\mathbb{P}_k^{r'-1}$ (s.(2.6)A)). In virtue of ((2.6)B) this shows that the modules

$$\operatorname{Tor}_i^{S'/\ell S'}(K, D(A)/\ell D(A)) \ , \ (i = 0, \ldots, r'-1) \ ,$$

are of the form given in statement a).

In order to prove b) and c), first note that the Castelnuovo-Mumford regularity $\mathrm{reg}(D(A))$ respectively takes the value 2 and 1 under the hypotheses of these two statements. In terms of [S$_1$] this means that $D(A)$ is an extremal Gorenstein resp. an extremal Cohen-Macaulay ring of size 2. Therefore the statements made in b) and c) are particular cases of Theorem B and Theorem A of [S$_1$] respectively. ■

In our next step we want to explore the homological structure of $D(A)$ as an S-module. We first make a preliminary observation on Koszul homology, which also will be useful later.

(5.2) Remark. Let $f = f_1\cdot,,\ldots,f_n$ and g be homogeneous elements of S'. For a graded S'-module M consider the Koszul homology modules $H_i(\underline{f}M)$ and $H_i(\underline{f},g;M)$ of M with respect to $\underline{f}$ respectively with respect to $\underline{f},g$. Then there is a graded long exact sequence

$$\ldots\ H_i(\underline{f};M)(-d) \xrightarrow{g} H_i(\underline{f};M) \to H_i(\underline{f},g;M) \to H_{i-1}(\underline{f};M)(-d)\ \ldots$$

where $d = \deg(g)$, see [Bru-He] for details. •

Now, we are ready to compute the graded Betti modules of the S-module $D(A)$. Note that the case $h^1(A)_1 = 0$ is already settled by (5.1)a), so that we may restrict our attention to the cases where $h^1(A)$ equals 1 or 2.

(5.3) Theorem. *Let $r \geq 3$.*

a) *If $h^1(A)_1 = 1$ (e.g. in the cases II and II′),*

$$\mathrm{Tor}_i^S\big(K, D(A)\big) \cong \begin{cases} K(0) \oplus K(-1) & \text{for } i = 0, \\ K^{e_i}(-i-1) & \text{for } 1 \leq i \leq r-2, \\ K(-r) \oplus K(-r-1) & \text{for } i = r-1, \end{cases}$$

where $e_i = r\binom{r-1}{i} - \binom{r-1}{i+1} - \binom{r-1}{i-1}$ *for* $i = 1,\ldots,r-2$.

b) *If $h^1(A)_1 = 2$ (e.g. in the cases III, IV, IV′ and IV″),*

$$\mathrm{Tor}_i^S\big(K, D(A)\big) \cong \begin{cases} K(0) \oplus K^2(-1) & \text{for } i = 0, \\ K^{f_i}(-i-1) & \text{for } 1 \leq i \leq r-1, \end{cases}$$

where $f_i = (r+1)\binom{r-1}{i} - \binom{r-1}{i+1}$ *for* $i = 1,\ldots,r-1$.

Proof. We write $D := D(A)$ and start with the proof of statement a). By our assumption we have $r' = r+1$. Now split the variables $\mathbf{x}_0,\ldots,\mathbf{x}_{r'}$ of S' into two subsets $\underline{\mathbf{x}} = \mathbf{x}_0,\ldots,\mathbf{x}_r$ and $\mathbf{y} = \mathbf{x}_{r+1}$. Then by (5.2) there is an exact sequence

$$\begin{aligned} & H_i(\underline{\mathbf{x}};D)(-1) \xrightarrow{\mathbf{y}_i} H_i(\underline{\mathbf{x}};D) \to H_i(\underline{\mathbf{x}},\mathbf{y};D) \\ & \qquad \longrightarrow H_{i-1}(\underline{\mathbf{x}};D)(-1) \xrightarrow{\mathbf{y}_{i-1}} H_{i-1}(\underline{\mathbf{x}};D) \longrightarrow H_{i-1}(\underline{\mathbf{x}},\mathbf{y};D) \end{aligned} \tag{$*$}$$

for all i, in which $\mathbf{y}_i$ denote the maps given by multiplication with $\mathbf{y}$ on $H_i(\underline{\mathbf{x}},D)(-1)$. First put $i = r$. Then, because of the depth sensitivity of Koszul complexes we know that $H_r(\underline{\mathbf{x}},D) = 0$. By $(*)$ we get an exact sequence

$$0 \to H_r(\underline{\mathbf{x}},\mathbf{y};D) \to H_{r-1}(\underline{\mathbf{x}};D)(-1) \xrightarrow{\mathbf{y}_{r-1}} H_{r-1}(\underline{\mathbf{x}};D) \to H_{r-1}(\underline{\mathbf{x}},\mathbf{y};D)\ .$$ •

By (5.1)b) we have $H_r(\underline{\mathbf{x}}, \mathbf{y}; D) \cong K(-r-2)$ and $H_{r-1}(\underline{\mathbf{x}}, \mathbf{y}; D) \cong K^{c_{r-1}}(-r)$. As $H_{r-1}(\underline{\mathbf{x}}; D)_n$ vanishes for all but finitely many $n \in \mathbb{Z}$ it follows that

$$H_{r-1}(\underline{\mathbf{x}}; D) \cong K(-r-1) \oplus K(-r) \ .$$

Moreover we see that coker $(\mathbf{y}_{r-1}) \cong K(-r)$. If we consider the sequence $(*)$ with $i = r-1$ we get an exact sequence

$$0 \to \ \text{coker } (\mathbf{y}_{r-1}) \to H_{r-1}(\underline{\mathbf{x}}, \mathbf{y}; D) \to H_{r-2}(\underline{\mathbf{x}}; D)(-1) \xrightarrow{\mathbf{y}_{r-2}} H_{r-2}(\underline{\mathbf{x}}; D) \to H_{r-2}(\underline{\mathbf{x}}, \mathbf{y}; D) \ .$$

By (5.1)b) we know that $H_{r-1}(\underline{\mathbf{x}}, \mathbf{y}; D) \cong K^{(c_{r-1})}(-r)$ and $H_{r-2}(\underline{\mathbf{x}}, \mathbf{y}; D) \cong K^{(c_{r-2})}(-r+1)$. By a similar argument as above it follows that

$$H_{r-2}(\underline{\mathbf{x}}, D) \ = \ K^{e_{r-2}}(-r+1) \quad \text{with} \quad e_{r-2} = c_{r-1} - 1 \ .$$

Moreover $\mathbf{y}_{r-2} : H_{r-2}(\underline{\mathbf{x}}; D)(-1) \to H_{r-2}(\underline{\mathbf{x}}, D)$ is the zero map. Now by descending induction on i and on use of (5.1)-b) it follows that

$$H_i(\underline{\mathbf{x}}, D) = K^{e_i}(-i-1) \quad \text{with} \quad e_i = c_{i+1} - e_{i+1}$$

and that $\mathbf{y}_i : H_i(\underline{\mathbf{x}}; D)(-1) \to H_i(\underline{\mathbf{x}}; D)$ is the zero map for $1 \le i \le r-2$. Finally, an easy calculation proves our claim for $i = 0$. Thus part a) is shown.

Next let us prove part b). By our assumption we have $r' = r+2$. Again, we split up the variables $\mathbf{x}_0, \ldots, \mathbf{x}_{r'}$ of S' and set $\underline{\mathbf{x}} = \mathbf{x}_0, \ldots, \mathbf{x}_{r+1}, \mathbf{y} = \mathbf{x}_{r+2}$. Once more, there is our exact sequence of Koszul homology modules as given in $(*)$. Our first aim is to show that

$$(**) \qquad H_i(\underline{\mathbf{x}}; D) \ \cong \ \begin{cases} K(0) \oplus K(-1) & \text{for } i = 0 \\ K^{d_i}(-i-1) & \text{for } 1 \le i \le r \ , \end{cases}$$

where $d_i = i\binom{r+1}{i+1} \ + \ \binom{r}{i} \ = \ (r+1)\binom{r}{i} \ - \ \binom{r+1}{i+1}, \quad (1 \le i \le r)$.

By (5.1)c) we have

$$H_{r+1}(\underline{\mathbf{x}}, \mathbf{y}; D) \cong K^{c_{r+1}}(-r-2) \quad \text{and} \quad H_r(\underline{\mathbf{x}}, \mathbf{y}; D) \cong K^{c_r}(-r-1) \ .$$

Moreover $H_{r+1}(\underline{\mathbf{x}}; D) = 0$. Now, the sequence $(*)$ with the choice $i = r+1$ gives rise to an exact sequence

$$0 \to H_{r+1}(\underline{\mathbf{x}}, \mathbf{y}; D)(-1) \to H_r(\underline{\mathbf{x}}, D) \xrightarrow{\mathbf{y}_r} H_r(\underline{\mathbf{x}}; D) \to H_r(\underline{\mathbf{x}}, \mathbf{y}; D)$$

which shows that

$$H_r(\underline{\mathbf{x}}; D) \cong K^{d_r}(-r-1) \quad \text{with} \quad d_r = c_{r+1}$$

and that

$$\mathbf{y}_r : H_r(\underline{\mathbf{x}}; D)(-1) \to H_r(\underline{\mathbf{x}}; D) \quad \text{is the zero map.}$$

Now, let $1 \le i < r$. By (5.1)c) we may write

$$H_{i+1}(\underline{\mathbf{x}}, \mathbf{y}; D) \cong K^{c_{i+1}}(-i-2) \quad \text{and} \quad H_i(\underline{\mathbf{x}}, \mathbf{y}; D) \cong K^{c_i}(-i-1) \ .$$

By $(*)$, we have an exact sequence

$$H_{i+1}(\underline{\mathbf{x}}; D)(-1) \xrightarrow{\mathbf{y}_{i+1}} H_{i+1}(\underline{\mathbf{x}}; D) \to H_{i+1}(\underline{\mathbf{x}}, \mathbf{y}; D) \to H_i(\underline{\mathbf{x}}; D)(-1) \xrightarrow{\mathbf{y}_i} H_i(\underline{\mathbf{x}}; D) \to H_i(\underline{\mathbf{x}}, \mathbf{y}; D) \ .$$

Altogether, this allows to conclude by descending induction on i, that

$$H_i(\underline{\mathbf{x}}; D) \cong K^{d_i}(-i-1) \quad \text{with} \quad d_i = c_{i+1} - d_{i+1}$$

and that

$$\mathbf{y}_i : H_i(\underline{\mathbf{x}}; D)(-1) \to H_i(\underline{\mathbf{x}}; D) \quad \text{is the zero map.}$$

In particular, this proves the claim made in $(**)$ for $1 \leq i \leq r$. Once more we now see by $(*)$, that $H_0(\underline{\mathbf{x}}, D)$ has the requested form. So, $(**)$ is shown.

In order to finish the proof of statement b), we split up the indeterminates $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{r+1}$ by setting $\underline{\mathbf{x}} = \mathbf{x}_0, \dots, \mathbf{x}_r$ and $\mathbf{y} = \mathbf{x}_{r+1},$. Note that now by $(**)$ the Koszul homology modules

$$H_i(\underline{\mathbf{x}}, \mathbf{y}; D) \ , \ 0 \leq i \leq r \ ,$$

are known. In this new context, (5.2) yields again an exact sequence $(*)$. Now, statement b) may be shown by following the outline of our previous arguments. ■

(5.4) Remark. A) Under the assumption of (5.3)a) (e.g. is the case I). It follows that a minimal free resolution $F_\bullet \to D(A)$ of the S-module $D(A)$ is self-dual, e.g. that $F_\bullet \cong \mathrm{Hom}_S\big(F_\bullet, S(-r-1)\big)$. In particular it turns out that

$$D(A) \cong \mathrm{Ext}_S^{r-1}\big(D(A), \ S(-r-1)\big) \ .$$

In other words, the $*$canonical module $\Omega_{D(A)}$ of the A-module $D(A)$ is isomorphic to $D(A)$, (s. [S$_2$]).

B) Let $F_\bullet \to D(A)$ be as above. This resolution and hence the Betti modules calculated in (5.1)a) and (5.3) a), b) provide information on the *Hilbert series* $F(t, D(A))$ of $D(A)$ (cf. [Bru-He, (4.1.13]). So, if $h^1(A)_1 = 2$ (e.g. in the cases III and IV), we conclude by (5.3)b) that

$$F\big(t, D(A)\big) \ = \ \frac{1}{(1-t)^{r+1}} \ \big(1 + 2t \ + \sum_{i=1}^{r-1} (-1)^i f_i t^{i+1}\big) \ .$$

On the other hand, the Hilbert series of $D(A)$ does not depend on whether we consider $D(A)$ as a module over S or over S'. Hence we may use (5.1)c) - or keep in mind that $D(A)$ is the homogeneous coordinate ring of a rational normal curve in $\mathbb{P}_K^{r+2}$ (s.(4.2)c)) - in order to conclude that

$$F(t, D(A)) \ = \ \frac{1}{(1-t)^2} \ \big(1 + (r+1)t\big) \ .$$

Comparing both expressions we see that

$$1 + 2t + \sum_{i+1}^{r-1} (-1)^i f_i t^{i+1} \ = \ \big(1 + (r+1)t\big)(1-t)^{r-1} \ .$$

This confirms again that $f_i = (r+1)\binom{r-1}{i+1} \ - \ \binom{r-1}{i}$ for $1 \leq i \leq r-1$.

If $h^1$231TfΩ57.999800TDΩ()TjΩ/TTDΩ31o161TfΩ32.9900(that)]TJΩ/T2151TfΩ48-30.0002T6(+1)TjΩ/

(5.5) Proposition. *The Hartshorne - Rao module* $H^1(A)$ *does not possess a graded direct summand isomorphic to* $K[\ell]/\ell^2[-1]$, *where* $\ell \in S_1 \backslash \{0\}$ *is a linear form.*

Proof. Assume the contrary. Then $H^1(A) \neq 0$ and A is not arithmatically Cohen-Macaulay. By (4.1)we know that $D := D(A)$ is a Cohen-Macaulay homogeneous K-algebra and the finite free resolution of D as an S-module is of the type described in (5.3). In particular it follows that $\mathrm{Tor}_1^S(K, D) \cong K^d(-2)$ in each of the possible cases. Now consider the short exact sequence

$$0 \to A \to D \to H^1(A) \to 0 \ ,$$

which induces an epimorphism

$$(* * *) \qquad \mathrm{Tor}_1^S(K, D) \to \mathrm{Tor}_1^S\big(K, H^1(A)\big) \to 0 \ .$$

To this end note that the induced homomorphism $A \otimes_S S/\underline{\mathbf{x}}S \to D \otimes_S S/\underline{\mathbf{x}}S$ is injective. The Koszul resolution of $K[\ell]/\ell^2[-1]$ shows that $K^r(-3)$ is a direct summand of $\mathrm{Tor}_1^S(K, H^1(A))$. This conradicts the previously observed isomorphism $\mathrm{Tor}_1^S(K, D) \cong K^d(-2)$ and the epimorphism $(* * *)$. ■

(5.6) Remark. Now clearly Proposition (5.5) yields the impossibility of the cases II′ and IV′ which was stated in (3.11): in both cases the Hartshorne-Rao module $H^1(A)$ would have a graded direct summand of the type excluded by (5.5). •

6. BETTI NUMBERS OF A

The aim of this section is to accomplish the (approximate) calculation of the Betti numbers of the homogeneous coordinate ring A of our curve $\mathcal{C} \subseteq \mathbb{P}^r$. Remember that by (5.1) a) we already know these Betti numbers in the case I. The case IV$''$, which only occurs if $r = 3$, need not be treated either (cf. [S_2]). So, it remains to treat the cases II, III, IV. We keep our previous notations and write S for the polynomial ring $K[\mathbf{x}_0, \ldots, \mathbf{x}_r]$ and A for the homogeneous coordinate ring of $\mathcal{C}$. One point of our interest shall be to decide in which cases $\mathcal{C}$ may lie on a surface of minimal degree. We thus start our considerations with a few preliminaries about curves on surfaces of minimal degree.

(6.1) Remark. A) (Cf. [Be, pg. 53/54]) Remember that an irreducible nondegenerate projective surface $X \subseteq \mathbb{P}^r_K$ always has degree $\geq r-1$ and that such a surface is said to be of *minimal degree* if it has degree $r-1$. A surface $X \subseteq \mathbb{P}^r$ of minimal degree is known to be either a *rational scroll* or a cone over a rational normal curve $Y \subseteq \mathbb{P}^{r-1}_K$ or - in the case $r = 5$ - a Veronese surface. By the so called $K_{p,1}$ - Theorem of Green, our curve $\mathcal{C} \subseteq \mathbb{P}^r_K$ lies on a surface $X \subseteq \mathbb{P}^r_K$ of minimal degree if and only if the $(r-1)$-homogeneous part $\mathrm{Tor}^S_{r-2}(K, A)_{r-1}$ of the graded S-module $\mathrm{Tor}^S_{r-2}(K, A)$ does not vanish. Moreover, in this case we have

$$\dim_K\left(\mathrm{Tor}^S_{r-2}(K, A)_{r-1}\right) \;=\; r-2$$

(see [G, 3.c.1]).

B) Remember, that rational scrolls are obtained as appropriate embeddings of geometrically ruled surfaces over $\mathbb{P}^1_K$ and that – up to isomorphism – each of this latter surfaces is isomorphic to a surface of the form

$$\mathbb{F}_a \;:=\; \mathbb{P}_{\mathbb{P}^1_K}\left(\mathcal{O}_{\mathbb{P}^1_K} \oplus \mathcal{O}_{\mathbb{P}^1_K}(a)\right) ,$$

with an uniquely determined number $a \in \mathbb{N}_0$ (s. [Be, III. 15, IV.1 (iii)]). Let $\pi : \mathbb{F}_a \to \mathbb{P}^1_K$ be the structural morphism and let $F, H \in \mathrm{Pic}(\mathbb{F}_a)$ denote the classes of the line bundles $\pi^*\left(\mathcal{O}_{\mathbb{P}^1_K}(1)\right)$ and $\mathcal{O}_{\mathbb{F}_a}(1)$ respectively, the latter being the tautological line bundle of $\mathbb{F}_a$ (s. [Be, III.17]). Then $\mathrm{Pic}(\mathbb{F}_a)$ is freely generated by F and H. For the intersection numbers involved with F and H we have $F \cdot F = 0$, $F \cdot H = 1$ and $H \cdot H = a$ (cf. [Be, IV.1 (i)]).

C) Keep the notation of part B). Then, the fibre $\pi^{-1}(p)$ of an arbitrary closed point $p \in \mathbb{P}^1_K$ is a projective line which belongs to the linear system $|F|$. So, if we apply the genus formula (cf. [Be, I.15]) to the curve $\pi^{-1}(p) \subseteq \mathbb{F}_a$ we see that the canonical divisor $\mathbb{K}$ of $\mathbb{F}_a$ satisfies $\mathbb{K} \cdot F = -2$. As $\mathbb{K} \cdot \mathbb{K} = 8$ (cf. [Be, III.21]), the observation made at the end of part B) shows that $\mathbb{K} = (a-2)F - 2H$.

Finally, if $D \subseteq \mathbb{F}_a$ is an irreducible curve defined by a divisor linearly equivalent to $\alpha F + \beta H (\alpha, \beta \in \mathbb{N}_0)$ another use of the genus formula shows that the arithmetic genus of D is given by

$$p_a(D) \;=\; a\binom{\beta}{2} + (\alpha - 1)(\beta - 1) .$$

D) Keep the notations of part B). Then if $a \geq 1$ the linear system $|H|$ defines a morphism $j_0 : \mathbb{F}_a \to \mathbb{P}^{a+1}_K$ which is such that $\mathbb{S}_{a,0} := j_0(\mathbb{F}_a) \subseteq \mathbb{P}^{a+1}_K$ is a singular surface of minimal degree, e.g. a cone over a rational normal curve (s. [Be, pg. 53]). Moreover, any singular surface of minimal degree in $\mathbb{P}^{a+1}_K$ can be obtained in this way.

For any $k \in \mathbb{N}$ and for any $a \geq 0$, the linear system $|H + kF|$ defines a closed immersion $j_k : \mathbb{F}_a \hookrightarrow \mathbb{P}^{a+2k+1}_K$ such that $j_k(\mathbb{F}_a) =: \mathbb{S}_{a,k} \subseteq \mathbb{P}^{a+2k+1}_K$ is a surface of minimal degree. Moreover, any non-singular surface of minimal degree – different from the Veronesean in $\mathbb{P}^5$ – can be obtained in this way (s. [Be, pg. 53/54]).

So, if we fix $r \geq 3$, the surfaces of minimal degree in $\mathbb{P}^r$ (different from the Veronesean surface in $\mathbb{P}^5$) are precisely the surfaces

$$\mathbb{M}_{r,k} := \mathbb{S}_{r-2k-1,k} \subseteq \mathbb{P}^r \quad \text{with} \quad 0 \leq k \leq \frac{r-1}{2} .$$

Moreover the surface $\mathbb{M}_{r,k}$ is singular if and only if $k = 0$ and $r \geq 3$.

E) The Veronese surface in $\mathbb{P}^5_K$ does not contain a curve of odd degree, so that our curve $\mathcal{C}$ does not lie on this surface if $r = 5$. So by the previous observation we see, that a minimal surface $X \subseteq \mathbb{P}^r$ which contains $\mathcal{C}$ must always be one of the surfaces $\mathbb{M}_{r,k}$. •

Now, using the above notations we have the following result.

(6.2) Lemma. *Let $r \geq 4$, $0 \leq k \leq \frac{r-1}{2}$, assume that $\mathcal{C} \subseteq \mathbb{M}_{r,k}$ and let $\alpha, \beta \in \mathbb{N}_0$ be such that $\mathcal{C}$ is given as a member of $|\alpha F + \beta H|$. Then $\beta \in \{1, 2, 3\}$. Moreover*

a) *if $\beta = 1$, then $p_a(\mathcal{C}) = 0$ and $\alpha = k + 3$;*

b) *if $\beta = 2$, then $p_a(\mathcal{C}) = 2, k \geq \frac{r-4}{2}$ and $\alpha = 2k - r + 4 \leq 3$;*

c) *if $\beta = 3$, then either: $p_a(\mathcal{C}) = 1, r = 4, k = 1, \alpha = 0$ or: $p_a(\mathcal{C}) = 0, r = 5, k = 2, \alpha = 1$.*

Proof. $r+2 = \deg(\mathcal{C}) = (kF + H)\cdot(\alpha F + \beta H) = k\alpha F \cdot F + (k\beta + \alpha)F \cdot H + \beta H \cdot H = k\beta + \alpha + \beta(r - 2k - 1)$. (see (6.1)B), D)) gives us the equation

$$(*) \qquad \alpha = (k + 1 - r)\beta + r + 2 .$$

By (6.1)C),D) we have

$$(**) \qquad p_a(\mathcal{C}) = (r - 2k - 1)\binom{\beta}{2} + (\alpha - 1)(\beta - 1) .$$

As $p_a(\mathcal{C}) \geq 0$ these equations immediately exclude the case $\beta = 0$. As $k \leq \frac{r-1}{2}$, we see from $(*)$ that

$$(\bullet) \qquad 0 \leq \alpha \leq \frac{1-r}{2}\beta + r + 2 .$$

In particular we see that $\beta \leq 2 + \frac{6}{r-1} \leq 4$. If $\beta = 4$ we must have $r = 4$ and the inequality $(\bullet)$ gives $\alpha = 0$, a contradiction to $(*)$. So we obtain $\beta \in \{1, 2, 3\}$.

Now, statement a) is obvious by $(*)$ and $(**)$. So, let $\beta = 2$. Then $(*)$ gives $\alpha = 2k - r + 4 \geq 0$ and $(**)$ gives $p_a(\mathcal{C}) = (r - 2k - 1) + \alpha - 1 = 2$, whereas $(\bullet)$ furnishes $\alpha \leq 3$.

If $\beta = 3$, $(*)$ shows that $\alpha = 3k - 2r + 5$ and $(**)$ gives $p_a(\mathcal{C}) = (r - 2k - 1)3 + (\alpha - 1)2 = 5 - r$, so that $r \in \{4, 5\}$ and $p_a(\mathcal{C}) \in \{0, 1\}$.

If $p_a(\mathcal{C}) = 1$, we have $r = 4$ and $(\bullet)$ gives $\alpha = 1$ so that finally $k = 1$.

If $p_a(\mathcal{C}) = 0$, we have $r = 5$ and $(\bullet)$ gives $\alpha \leq 1$. In this case, we also have $\alpha = 3k - 5$, hence $\alpha \neq 0$ so that $\alpha = 1$ and $k = 2$. ■

(6.3) Remark. A) If $h^1(A)_1 = 0$, e.g. in the case I, (5.1)a) and the observation made in (6.1)A) show, that $\mathcal{C}$ is contained in some surface $X \subseteq \mathbb{P}^r_K$ of minimal degree. Now, by (6.1)D), E) and by (6.2) and by (3.9) we see that X is non-singular whenever $r \geq 5$.

B) In the case II we have $p_a(\mathcal{C}) = 1$ (see (3.9)). So, if in this case, the curve $\mathcal{C}$ is contained in a surface $X \subseteq \mathbb{P}^r$ of minimal degree, we see by (6.1)E) and by (6.2) that $r \leq 4$. •

Part B) of the previous remarks will be used later to show that the module $\mathrm{Tor}^S_{r-2}(K, A)_{r-1}$ vanishes in the case II if $r \geq 5$. Next, we prove another result which also concerns the case II for $r \geq 5$.

(6.4) Lemma. *Let $h^1(A)_1 = 1$ and $r \geq 5$. Then, the defining ideal $I \subseteq S$ of $\mathcal{C}$ is generated by quadrics and at most one cubic.*

Proof: We are in the case II, so that $h^1(A)_2 = 0$ and $h^2(A)_1 = 0$. So A is 2-regular and hence I is 3-regular and thus may be generated by quadrics and cubics. So, we write $I = (J, L)$ with $J = I_2 S$ and with a K-vector space $L \subseteq S_3$ such that $I_3 = J_3 \oplus L$. Our aim is to show that $\dim_K(L) \leq 1$.

After an appropriate linear change of coordinates we may assume that $\mathbf{x}_r \in S_1$ is generic with respect to $\mathcal{C}$ in the sense of (2.6)A). Let $T := S/\mathbf{x}_r S \cong K[\mathbf{x}_0, \ldots, \mathbf{x}_{r-1}]$. Then $R := T/(J,L)T \cong S/(I, \mathbf{x}_r)S$ defines a scheme X of $r+2$ points in semiuniform position in $\mathbb{P}^{r-1}_K$. Now, the graded short exact sequence $0 \to A(-1) \xrightarrow{\mathbf{x}_r} A \to R \to 0$ and the isomorphism $H^1(A) \cong K(-1)$ show that $H^0(R) \cong K(-2)$. But this means that the vanishing ideal of X in T has the form $(J, L, q)T$ with an appropriate quadric $q \in S_2$. As $r \geq 5$, the minimal free resolution of this ideal gives us a graded exact sequence

$$(*) \qquad T^{a_2}(-3) \xrightarrow{\varphi} T^{a_1}(-2) \xrightarrow{\pi} (J, L, q)T \to 0$$

(see (2.6)B)). This allows to write $(J, L, q)T = (J, q)T$ and to assume that the first $a_1 - 1$ canonical basic elements of $T^{a_1}(-2)$ are mapped by π onto a K-basis of $(JT)_2$ and that the last canonical basis element of $T^{a_1}(-2)$ is mapped to $q \cdot 1_T$. Clearly, the map φ is given by a matrix whose entries are linear forms in T. This shows that $M := (JT :_T q) \subseteq T$ is a proper ideal generated by linear forms:

As $JT \subseteq M$ and as $(J, q)T = IT$ is of height $r-1$ we must have $r - 2 \leq \text{height}(M) \leq r$. As M is generated by linear forms $(T/M)_1$ is a K-vector space of dimension $t := r - \text{height}(M) \in \{0, 1, 2\}$. So, the graded short exact sequence

$$0 \to T/M(-2) \to T/JT \to T/(J,q)T \to 0$$

shows that $\dim_K(IT)_3 = \dim_K\big((J,q)\big)_3 = \dim_K(JT)_3 + t$. So, we may write $(I, \mathbf{x}_r) = (J, L', \mathbf{x}_r)$, where $L' \subseteq L \subseteq S_3$ is K-subspace of dimension $\leq t$. As I is prime and as $\mathbf{x}_r \in S_1 \backslash I$ it follows $I = (J, L')$, hence $L' = L$. So, if $\dim_K(L') \leq 1$, we are done.

Otherwise we have $\dim_K(L) = \dim_K(L') = 2 = t$ and we may write $I = (J, k_1, k_2)$ with $k_1, k_2 \in S_3$. As height $(I) = r - 1$, it follows height $(J) \geq r - 3$. As $JT \subseteq M$ and as height $(M) \leq r - 2$, we have height $(JT) \leq r - 2$. As $\mathbf{x}_r$ was a generic linear form, this means that height $(J) \leq r - 3$ and hence height $(J) = r-3$. As $I = (J, k_1, k_2)$ is a prime of height $r-1 = \text{height}(J)+2$, J must be prime, too. As $\mathbf{x}_r$ was generic, we may conclude by Bertini that $JT \subseteq T$ defines an integral subscheme of $\mathbb{P}^{r-1}$. So, the saturation $JT : \langle T_+ \rangle \subseteq T$ of JT in T is a prime ideal of height $r-2$. As $JT \subseteq M \subsetneqq T_+$ and as M is a prime ideal of height $r-2$, we get $JT : \langle T_+ \rangle = M$. So, $\text{Proj}(T/IT) = \text{Proj}(T/(J,q)T) = \text{Proj}\big(T/(JT,\ qT)\big) = \text{Proj}\big(T/(M, qT)\big)$ consists of two points so that $r + 2 = 2$ - a contradiction. So, this case does not occur at all. ■

Before we attack the calculation of the Betti numbers of A, we make a general remark which explains our further strategy and paves the way to our calculations.

(6.5) Remark. A) Keep all previous notations and let $D := D(A)$ and $H := H^1(A)$. In chapter 5 we have calculated the Betti numbers of the S-module D. Moreover we know the structure of the Hartshorne-Rao module H in each of the occuring cases (see (3.9), (3.10)), so that we can calculate its Betti numbers. Hence there is much evidence, that the short exact sequence

$$(*) \qquad 0 \to A \to D \to H \to 0$$

allows to approximate the Betti numbers of A in a satisfactory way.

B) Once more, let $\ell \in S_1$ be a generic linear form and let $T := S/\ell S$. Then, by what is said in (3.9) on the structure of H, ℓ acts surjectively on H, so that $H^0(A/\ell A) \cong (0 :_H \ell)(-1)$. As a consequence we get the short exact sequence of graded T-modules

$$(**) \qquad 0 \to (0 :_H \ell)(-1) \to A/\ell A \to B \to 0$$

where $B := (A/\ell A)/H^0(A/\ell A)$. As B defines a scheme of $r+2$ points in semi-uniform position in $\mathbb{P}^{r-1}_K = \mathrm{Proj}(T)$, we know the Betti numbers of the T-module B (see (2.6)B)). Moreover we can calculate the Betti numbers of the T-module $0{:}_H\ell$ and hence use the sequence $(**)$ to approximate the Betti numbers of the T-module $A/\ell A$. As ℓ is at the same time A- and S-regular, these latter Betti numbers coincide with the Betti numbers of the S-module A.

C) In order to relate the approaches sketched in A) and B), we make use of the following result: Let $0 \to M \to N \to P \to 0$ be an exact sequence of graded S-modules. Let $n \in \mathbb{N}$. Then:

(i) If $P_n = 0$ (resp. $P_n = P_{n-1} = 0$), the induced map $\mathrm{Tor}^S_p(K,M)_{p+n} \to \mathrm{Tor}^S_p(K,N)_{p+n}$ is surjective (resp. bijective) for all $p \in \mathbb{N}$.

(ii) If $M_n = 0$ (resp. $M_n = M_{n+1} = 0$), the induced map $\mathrm{Tor}^S_p(K,N)_{p+n} \longrightarrow \mathrm{Tor}^S_p(K,P)_{p+n}$ is injective (resp. bijective) for all $p \in \mathbb{N}$.

In order to prove statement (i), observe that $\mathrm{Tor}^S_p(K,P)_{p+n} \cong H_p(\mathbf{x}_0,\ldots,\mathbf{x}_r;P)_{p+n}$ is a subquotient of $\left[P^{\binom{r+1}{p}}(-p)\right]_{p+n} = 0$. Statement (ii) follows similarly. •

Now, we give the announced approximation of the Betti numbers of A in the remaining cases II, III and IV. We start with the case II.

(6.6) Theorem. *Let $r \geq 3$. Assume that $h^1(A)_1 = 1$, so that we are in the case II. Then*

$$\mathrm{Tor}^S_i(K,A) \cong K^{u_i}(-i-1) \oplus K^{v_i}(-i-2),\ (i = 1,\ldots,r)\ ,$$

where the Betti numbers u_i and v_i are given (respectively bounded) according to the following table

i	1	$2 \leq i \leq r-3$	$r\ -\ 2$	$r-1$	r
u_i	$\binom{r}{2}-3$	$\leq\quad a_i$	u_{r-2}	0	0
v_i	v_1	$\leq\quad \binom{r}{i}$	$\binom{r+1}{2}-1$	$r+2$	1

Moreover

a) *If $r \geq 5$, then $v_1 \leq 1$;*

b) *$u_{r-2} \in \{0, r-2\}$; with $u_{r-2} = 0$ if $r \geq 5$;*

c) *$a_i = i\binom{r}{i+1} - 2\binom{r-1}{i-1}$ for $2 \leq i \leq r-3$.*

d) *$u_i - v_{i-1} = a_i - \binom{r}{i-1}$ for $2 \leq i \leq r-2$.*

Proof. By our assumption, A is 2-regular and of depth 1. Therefore the modules $\mathrm{Tor}^S_i(K,A)$ have the requested form in the range $1 \leq i \leq r$ and vanish for $i > r$. Moreover $H^1(A) \cong K(-1)$. So, in the notations of (6.5)B) we have $(0 :_H \ell)(-1) \cong K(-2)$ and hence

$$\mathrm{Tor}^T_i\left(K, (0 :_H \ell)(-1)\right) \cong K^{\binom{r}{i}}(-i-2) \quad \text{for} \quad i = 0,\ldots,r\ .$$

So, the $\operatorname{Tor}^T_\bullet(K,\bullet)$ -sequence associated to the sequence (**) of (6.5)B) gives rise to exact sequences

$$0 \to \operatorname{Tor}^T_i(K, A/\ell A)_{i+1} \to \operatorname{Tor}^T_i(K,B)_{i+1} \to K^{\binom{r}{i-1}} \to \operatorname{Tor}^T_{i-1}(K, A/\ell A)_{i+1} \to \operatorname{Tor}^T_{i-1}(K,B)_{i+1} \to 0$$

for all $i \in \mathbb{N}$. Moreover, as proposed in (6.5)B) we may use (2.6)B) to write

$$\operatorname{Tor}^T_i(K,B) \cong \begin{cases} K^{a_i}(-i-1) & \text{for } \ 1 \leq i \leq r-3 \ , \\ K^{r-1}(-r) \oplus K^{r-2}(-r+1) & \text{for } \ i = r-2 \ , \\ K^2(-r-1) & \text{for } \ i = r-1 \ , \end{cases}$$

$$\text{where} \quad a_i = i\binom{r}{i+1} - 2\binom{r-1}{i-1} \quad \text{for} \quad 1 \leq i \leq r-3 \ .$$

As $\operatorname{Tor}^S_i(K,A) \cong \operatorname{Tor}^T_i(K, A/\ell A)$ (cf.(6.5)B)), this gives what is said in the table and in statement c) on the values u_i for $i \neq r-2$, what is said on the values v_i for $i > 1$, and what is said in statement d). Statement a) is a consequence of (6.4). Finally by (6.3)B) the curve $\mathcal{C}$ is not contained in a surface $X \subseteq \mathbb{P}^r$ of minimal degree if $r \geq 5$. So, statement b) follows by (6.1)A). ■

Now, let us illustrate Theorem (6.6) by some examples.

(6.7) Examples. A) Consider the two non-degenerate rational curves $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{P}^7_K$ of degree 9 which are parametrically given by

$$\mathcal{C}_1 \ : \ (s^9, s^7t^2, s^6t^3, s^5t^4, s^4t^5, s^3t^6, st^8, t^9) \ ,$$

$$\mathcal{C}_2 \ : \ (s^9, s^7t^2, s^6t^3, s^5t^4, s^4t^5, s^2t^7, st^8, t^9) \ .$$

Both curves are obtained by projecting a rational normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}^9_K$ from a line and both have a unique double point (at $p = (1:0:\ldots:0)$). So, by (4.3)c) and (4.6) both curves $\mathcal{C}_j$ ($j = 1,2$) go under case II_1. The corresponding Betti numbers u_i and v_i are as listed in the following table

j	i	1	2	3	4	5	6	7
1	u_i	18	52	60	24	0	0	0
	v_i	1	6	15	30	27	9	1
2	u_i	18	51	55	18	0	0	0
	v_i	0	1	9	30	27	9	1

In particular, the curve $\mathcal{C}_2$ shows that v_1 may vanish.

B) If $r \leq 4$, it may happen that $v_1 > 1$. To see this, consider the non-degenerate rational curve $\mathcal{C} \subseteq \mathbb{P}^4_K$ of degree 6 parametrically given by

$$(s^6, s^4t^2, s^3t^3, st^5, t^6)$$

Again $\mathcal{C}$ is obtained by projecting a rational normal curve $\tilde{\mathcal{C}} \subseteq \mathbb{P}^6_K$ from a line and has a unique double point so that we are again in the case II_1. The Betti numbers are as listed below.

i	1	2	3	0
u_i	3	2	0	0
v_i	4	9	6	1

So, to define $\mathcal{C}$, we need 4 cubics (and 3 quadrics). As $u_2 = 2 \neq 0$, $\mathcal{C}$ is contained in a surface $X \subseteq \mathbb{P}^4_K$ of minimal degree (see (6.1)A)). Indeed, the quadrics $\mathbf{x}_1\mathbf{x}_4 - \mathbf{x}_2\mathbf{x}_3$, $\mathbf{x}_0\mathbf{x}_4 - \mathbf{x}_2^2, \mathbf{x}_0\mathbf{x}_3 - \mathbf{x}_1\mathbf{x}_2$ define such a surface.

C) Let $\mathcal{C}_0 \subseteq \mathbb{P}^2_K$ be a non-singular plane cubic. Let $n \in \mathbb{N}$, let $N = \binom{n+2}{2} - 1$ and let $v^{(n)} : \mathbb{P}^2_K \to \mathbb{P}^N_K$ be the n-th Veronese embedding. Let $\tilde{\mathcal{C}} := v^{(n)}(\mathcal{C}_0) \subseteq \mathbb{P}^N$ so that $\tilde{\mathcal{C}}$ is an elliptic curve of degree $3n$. Moreover, the linear hull $\mathbb{P}^{r+1}_K$ of $\tilde{\mathcal{C}}$ in $\mathbb{P}^N$ is of dimension $r + 1 = \dim_K\big(\Gamma(\mathcal{C}_0), \mathcal{O}_{\mathcal{C}_0}(n)\big) - 1 = 3n - 1$. So with $r = 3n - 2$ we see that $\tilde{\mathcal{C}} \subseteq \mathbb{P}^{r+1}_K$ is an elliptic normal curve. Hence if we project $\tilde{\mathcal{C}}$ from a generic point in $\mathbb{P}^{r+1}_K$ we get a curve $\mathcal{C} \subseteq \mathbb{P}^r_K$ which falls under case II_0, (cf. (4.7)(i)).

To present a concrete example, we assume that K is of characteristic $\neq 3$ and that $\mathcal{C}_0 \subseteq \mathbb{P}^2_K$ is defined by the cubic form $\mathbf{x}^3 + \mathbf{y}^3 + \mathbf{z}^3 \in K[\mathbf{x}, \mathbf{y}, \mathbf{z}]$. Then , we choose $n = 2$, so that $\tilde{\mathcal{C}} = v^{(2)}(\mathcal{C}_0)$ becomes an elliptic normal curve in $\mathbb{P}^5_K$. Now, let $\mathcal{C}_1, \mathcal{C}_2 \subseteq \mathbb{P}^4_K$ be the curves of degree 6 which are obtained by projecting $\tilde{\mathcal{C}}$ respectively from the points

$$p_1 := (0:0:0:0:0:1) = v^{(2)}(0:0:1) \notin v^{(2)}(\mathcal{C}^0) = \tilde{\mathcal{C}}$$

$$p_2 := (0:0:0:0:1:0) \notin v^{(2)}(\mathbb{P}^2_K) \supseteq \tilde{\mathcal{C}}$$

So the two curves $\mathcal{C}_j (j = 1, 2)$ fall under case I_1 or II_0 (s. (4.7)). The corresponding Betti numbers are as listed below.

j	i	1	2	3	4
1	u_i	3	2	0	0
	v_i	4	9	6	1
2	u_i	3	0	0	0
	v_i	2	9	6	1

In particular we see that none of the two curves $\mathcal{C}_j$ is arithmetically CM, so that both must fall under case II_0 (s. (3.9)). Moreover $\mathcal{C}_1$ is contained in a surface $X \subseteq \mathbb{P}^4_K$ of minimal degree (s. (6.1)A)) and the three quadrics $\mathbf{x}_0\mathbf{x}_3 - \mathbf{x}_1^2, \mathbf{x}_1\mathbf{x}_2 - \mathbf{x}_0\mathbf{x}_3$, $\mathbf{x}_1\mathbf{x}_4 - \mathbf{x}_2\mathbf{x}_3$ define indeed such a surface X containing $\mathcal{C}_1$.

Keep $\mathcal{C}_0$ as above and choose $n = 3$, so that $v^{(3)}(\mathcal{C}_0) = \tilde{\mathcal{C}} \subseteq \mathbb{P}^8_K$ is an elliptic normal curve. If we project respectively from the points

$$p_1 = (0:0:0:0:1:0:\cdots:0) \in \mathbb{P}^8 \backslash v^{(3)}(\mathbb{P}^2)$$

$$p_2 = (1:0:\cdots:0) \in \mathbb{P}^8 \backslash \tilde{\mathcal{C}}$$

we get two curves $\mathcal{C}_j \in \mathbb{P}^7$ $(j = 1, 2)$ of degree 9. The corresponding Betti numbers are as follows.

j	i	1	2	3	4	5	6	7
1	u_i	18	52	60	24	0	0	0
	v_i	1	6	15	30	27	9	1
2	u_i	18	51	54	18	0	0	0
	v_i	0	0	9	30	27	9	1

As previously we now recognize both curves $\mathcal{C}_1, \mathcal{C}_2$ to go under case II_0. Observe in particular, that $\mathcal{C}_2$ has $v_1 = v_2 = 0$. •

(6.8) Remark. Assume that we are still in the case II that $r \geq 5$ and that our curve has a trisecant line $\mathbb{P}^1 \subseteq \mathbb{P}^r_K$ which is defined by a linear ideal $L \subseteq S$. In this case, we have $v_1 = 1$ (cf. (6.6)) so that the defining ideal $I \subseteq S$ of $\mathcal{C}$ may be written in the form $I = (J, k)$ where $J \subseteq S$ is an ideal generated by quadrics and where $k \in S$ is a cubic.

In order to see this, observe that $S/(L, I)$ defines a subscheme of length 3 in $\mathrm{Proj}(S/L) = \mathbb{P}^1$. So, we must have $I_2 \subseteq L$. As I is generated by quadrics and cubics, there also must be a cubic $k \in I \backslash L$. As $I_2 S \subseteq L$, it follows $v_1 \geq 1$. By (6.4) we also have $v_1 \leq 1$, thus $v_1 = 1$.

Our next claim is, that in the above notation, we have $J :_S k = L$.

By (6.6) we know indeed that $I = (J, k)$ has a minimal free resolution beginning with

$$S^{u_2}(-3) \oplus S^{v_1}(-4) \to S^{u_1}(-2) \oplus S(-3) \to (J, k) \to 0 \ .$$

As in the proof of (6.4) we may conclude that $M := (J :_S k)$ is an ideal generated by linear forms. As $J = I_2 S \subseteq L$ and $k \notin L$ we have $J \subseteq J :_S k = M \subseteq L :_S k = L$. In particular $\mathrm{height}(J) \leq \mathrm{height}(L) = r - 1$. Assume that $\mathrm{height}(J) < r - 1$. Then $(J, k) = I \in \mathrm{Spec}(S)$ and $\mathrm{height}(I) = r - 1$ show that J is prime and this leads to the contradiction that $M = J :_S k = J$. Therefore, $\mathrm{height}(J) = r - 1$ and hence $M = J : k = L$. •

Now, we consider the case III.

(6.9) Theorem. *Let $r \geq 3$ and assume that $\mathcal{C}$ is of numerical cohomology type $(2, 0)$, so that we are in the case III. Then*

$$\mathrm{Tor}^S_i(K, A) \cong K^u(-i-1) \oplus K^{v_i}(-i-2), \ (i = 1, \dots, r) \ ,$$

where the Betti numbers u_i and v_i are given (respectively bounded) according to the following table

i	1	$2 \leq i \leq r-3$	$r-2$	$r-1$	r
u_i	$\binom{r}{2} - 4$	$\leq \quad a_i$	u_{r-2}	0	0
v_i	v_1	$\leq \quad 2\binom{r}{i}$	$r^2 - 1$	$2r + 2$	2

Moreover

a) $u_{r-2} \in \{0,\ r-2\}$ *with* $u_{r-2} = 0$ *if and only if* $\mathcal{C}$ *is not contained in a surface* $X \subseteq \mathbb{P}^r_K$ *of minimal degree;*

b) $a_i = i\binom{r}{i+1} - 2\binom{r-1}{i-1}$ *for* $2 \leq i \leq r-3$;

c) $u_i - v_{i-1} = a_i - 2\binom{r}{i-1}$ *for* $2 \leq i \leq r-2$.

Proof: By our assumption, A is 2-regular and of depth 1. Therefore the modules $\operatorname{Tor}_i^S(K, A)$ have the requested form in the range $1 \leq i \leq r$ and vanish for $i > 1$. Moreover $H^1(A) \cong K^2(-1)$. So, in the notations of (6.5)B) we have $(0 :_H \ell)(-1) \cong K^2(-2)$ and hence

$$\operatorname{Tor}_i^T\big(K, (0 :_H \ell)(-1)\big) \cong K^{2\binom{r}{i}}(-i-2) \quad \text{for} \quad i = 0, \ldots, r\ .$$

Now, we can prove what is said in the table and in statement b) on the values u_i for $i \neq r-2$, what is said in the table on the values v_i and what is said in statement c) by the same argument as in the proof of (6.6). Statement a) follows again from Green's $K_{p,1}$-Theorem (cf. (6.1)A)). ■

Let us consider a few examples.

(6.10) Examples. A) Consider the five non-degenerate rational curves $\mathcal{C}_j \subseteq \mathbb{P}^7_K$ of degree 9 ($j = 1, \ldots, 5$) parametically given respectively by

$$\begin{aligned}
\mathcal{C}_1 &: \big(s^9, s^8t, s^6t^3, s^4t^5, s^3t^6, s^2t^7, st^8, t^9\big)\ ,\\
\mathcal{C}_2 &: \big(s^9, s^8t, s^6t^3, s^5t^4, s^4t^5, s^2t^7, st^8, t^9\big)\ ,\\
\mathcal{C}_3 &: \big(s^9, s^8t, s^7t^2, s^4t^5, s^3t^6, s^2t^7, st^8, t^9\big)\ ,\\
\mathcal{C}_4 &: \big(s^9, s^8t, s^7t^2, s^5t^4, s^3t^6, s^2t^7, st^8, t^9\big)\ ,\\
\mathcal{C}_5 &: \big(s^9, s^8t, s^7t^2, s^6t^3, s^3t^6, s^2t^7, st^8, t^9\big)\ ,
\end{aligned}$$

Fix $j \in \{1, \ldots, 5\}$. Then, the defining monomials of $\mathcal{C}_j$ span a K-space of dimension 8 and each monomial $s^{18-k}t^k$ of degree 18 can be written as a product of two defining monomials of $\mathcal{C}_j$. This shows that $\mathcal{C}_j$ is of numerical cohomology type $(2, 0)$ and thus goes under case III. The Betti numbers u_i and v_i of the five curves $\mathcal{C}_j$ are as listed below.

j	i	1	2	3	4	5	6	7
1	u_i	17	46	45	8	0	0	0
	v_i	2	12	34	65	48	16	2
2	u_i	17	45	45	14	0	0	0
	v_i	1	12	40	65	48	16	2
3	u_i	17	46	45	24	5	0	0
	v_i	2	12	50	70	48	16	2
4	u_i	17	44	36	8	0	0	0
	v_i	0	3	34	65	48	16	2
5	u_i	17	44	45	24	5	0	0
	v_i	0	12	50	70	48	16	2

B) We do not know whether for $r \geq 6$ the defining ideal $I \subseteq S$ of the curve $\mathcal{C}$ needs at most two cubic generators in the case III. The following example shows, that this fails for $r = 5$. Namely, let $\mathcal{C} \subseteq \mathbb{P}^5_K$ be the non-degenerate rational curve of degree 7 parametrically given by

$$\left(s^7, s^6t, s^5t^2, s^2t^5, st^6, t^7\right) .$$

As in the examples given in A), we see that $\mathcal{C}$ falls under the case III. Here, the Betti numbers are as follows.

i	1	2	3	4	5
u_i	6	8	3	0	0
v_i	6	20	24	12	2

Moreover, the six quadrics

$$\mathbf{x}_0\mathbf{x}_2 - \mathbf{x}_1^2 \ , \quad \mathbf{x}_0\mathbf{x}_4 - \mathbf{x}_1\mathbf{x}_3 \ , \mathbf{x}_0\mathbf{x}_5 - \mathbf{x}_2\mathbf{x}_3$$
$$\mathbf{x}_0\mathbf{x}_5 - \mathbf{x}_1\mathbf{x}_4 \ , \mathbf{x}_1\mathbf{x}_5 - \mathbf{x}_2\mathbf{x}_4 \ , \mathbf{x}_3\mathbf{x}_5 - \mathbf{x}_4^2$$

define a surface $X \subseteq \mathbb{P}^5_K$ of minimal degree which contains $\mathcal{C}$.

C) Choose $r \geq 3$ arbitrarily and let $\mathcal{C} \subseteq \mathbb{P}^r_K$ the non-degenerate rational curve of degree $r + 2$ given parametrically by

$$\left(s^{r+2}, s^{r+1}t, \ldots, s^5t^{r-3}, s^2t^r, st^{r+1}, t^{r+2}\right) .$$

As above one sees that $\mathcal{C}$ falls under the case III. Now, let $X \subseteq \mathbb{P}^r_K$ be the rational surface scroll defined by the 2×2 -minors of the matrix

$$\begin{bmatrix} \mathbf{x}_0 & \mathbf{x}_1 & \cdots & \mathbf{x}_{r-4} & \mathbf{x}_{r-2} & \mathbf{x}_{r-1} \\ \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_{r-3} & \mathbf{x}_{r-1} & \mathbf{x}_r \end{bmatrix}$$

Observe that X is a surface of minimal degree (cf. [Be]) and that $\mathcal{C} \subseteq X$. So, in contrast to what happens in the case II, in the case III the curve $\mathcal{C}$ can be contained in a surface $X \subseteq \mathbb{P}^r_K$ of minimal degree, for any choice of r. •

Now, we look at the case where $\mathcal{C}$ belongs to the class IV thus to the case of n.c.t. (2.1). We begin with an auxiliary result about the Hartshorne-Rao module

$$H = H^1(A) \cong \operatorname{Hom}_K\big(S/((\mathbf{x}_0, \mathbf{x}_1)^2, \mathbf{x}_2, \cdots, \mathbf{x}_r)S, K\big)\ (-2)$$

of our curve $\mathcal{C}$ (cf. (3.10)). To formulate this result, let $\ell \in S_1$ be a generic linear form and let $T := S/\ell S$. Without loss of generality we may assume that $\ell = \mathbf{x}_0$.

(6.11) Lemma.

a) *For the S-module H we have*

$$\operatorname{Tor}_i^S(K, H) \cong K^{s_i}(-i-1) \oplus K^{t_i}(-i-2),\ (1 \leq i \leq r+1)\ ,$$

where $s_i = 3\binom{r-1}{i-1} + 2\binom{r-1}{i}$ *and* $t_i = \binom{r-1}{i-2}$.

b) *For the T-module* $(0 :_H \ell)(-1)$ *we have*

$$\operatorname{Tor}_i^T\big(K, (0 :_H \ell)(-1)\big) \cong K^{\overline{s_i}}(-i-3) \oplus K^{\overline{t_i}}(-i-2),\ (1 \leq i \leq r)\ ,$$

where $\overline{s_i} = \binom{r-1}{i-1}$ *and* $\overline{t_i} = \binom{r-1}{i}$.

Proof: "a)": First observe that $\operatorname{Tor}_i^S(K, H) \cong H_i(\underline{\mathbf{x}}_i; H)$ for all $i \in \mathbb{N}$, where $\underline{\mathbf{x}} := \mathbf{x}_0, \ldots, \mathbf{x}_r$. We write

$$H = \operatorname{Hom}_K\big(S/((\mathbf{x}_0, \mathbf{x}_1)^2, \mathbf{x}_2, \cdots, \mathbf{x}_r)S, K\big)(-2)$$

and prove our claim by induction on r. If $r = 1, H$ has a minimal graded free resolution of the shape

$$0 \to S(-4) \to S(-2)^3 \to S(-1)^2 \to H \to 0$$

and our claim is immediately clear.

If $r > 1$, we write $\underline{\mathbf{x}}' =: \mathbf{x}_0, \ldots, \mathbf{x}_{r-1}$ and $\mathbf{x} = \mathbf{x}_r$. As $\mathbf{x}H = 0$, (5.2) gives us isomorphisms

$$H_i(\underline{\mathbf{x}}; H) \cong H_i(\underline{\mathbf{x}}'; H) \oplus H_{i-1}(\underline{\mathbf{x}}'; H)(-1)\ , \quad (\forall i \in \mathbb{Z})\ ,$$

which allows to complete the induction.

"b)": Choosing $\ell = \mathbf{x}_0$, we have a graded isomorphism

$$(0 :_H \ell)(-1) \cong \operatorname{Hom}_K\big(T/(\mathbf{x}_1^2, \mathbf{x}_2, \ldots, x_r)\ T, K\big)(-3)$$

which allows to prove our claim by induction on r by similar arguments as in the proof of a). ■

(6.12) Theorem. *Let $r \geq 3$ and assume that $\mathcal{C}$ is of numerical cohomology type $(2,1)$, so that we are in the case IV. Then*

$$\operatorname{Tor}_i^S(K,A) \cong K^{u_i}(-i-1) \oplus K^{v_i}(-i-2) \oplus K^{v_i}(-i-3), \quad (i = 1, \ldots, r) \ ,$$

where the Betti numbers u_i, v_i and w_i are given (respectively bounded) according to the following table

i	1	$2 \leq i \leq r-3$	$r-2$	$r-1$	r
u_i	$\binom{r}{2} - 3$	$\leq \quad a_i$	u_{r-2}	0	0
v_i	$v_1 \leq r-1$	$\leq \quad \binom{r-1}{i}$	$2r-2$	3	0
w_i	1	$\binom{r-1}{i-1}$	$\binom{r-1}{2}$	$r-1$	1

Moreover

a) $u_{r-2} \in \{0, r-2\}$ *with* $u_{r-2} = 0$ *if and only if $\mathcal{C}$ is not contained in a surface $X \subseteq \mathbb{P}_K^r$ of minimal degree;*

b) $a_i = i\binom{r}{i+1} - 2\binom{r-1}{i-1}$ *for* $2 \leq i \leq r-3$;

c) $u_i - v_{i-1} = (r-1)\binom{r-1}{i} - \binom{r-1}{i+1} - 3\binom{r-1}{i-1}$ *for* $2 \leq i \leq r-2$.

Proof: By our assumption A is 3-regular and of depth 1, so that the modules $\operatorname{Tor}_i^S(K,A)$ have the requested form and vanish for $i > r$. Let $H := H^1(A)$, $D := D(A)$. Then, the short exact sequence $(*)$ of (6.5)A) gives rise to graded exact sequences

$$(\bullet) \qquad \operatorname{Tor}_{i+1}^S(K,D) \to \operatorname{Tor}_{i+1}^S(K,H) \to \operatorname{Tor}_i^S(K,A) \to \operatorname{Tor}_i^S(K,D) \to \operatorname{Tor}_i^S(K,H)$$

for all $i \in \mathbb{Z}$. W

$$(\bullet\bullet)'' \qquad \operatorname{Tor}_i^T(K,B) \;\cong\; \begin{cases} K^{a_i}(-i-1) & \text{for } 1 \le i \le r-3\ , \\ K^{r-1}(-r) \oplus K^{r-2}(-r+1) & \text{for } i = r-2\ , \\ K^2(-r-1) & \text{for } i = r-1\ , \end{cases}$$

$$(\bullet\bullet)''' \qquad \operatorname{Tor}_i^S(K,A) \;\cong\; \operatorname{Tor}_i^T(K, A/\ell A)\ ,$$

where the numbers $\overline{s_i}$ and $\overline{t_i}$ are defined according to (6.11)b) and the numbers a_i are defined as in (2.6)B). If we apply $(\bullet\bullet)$ with $i = r-1$ we first get $u_{r-1} = 0$ and hence $\operatorname{Tor}_{r-1}^S(K,A) \;\cong\; K^3(-r-1) \oplus K^{r-1}(-r-2)$. Now, apply $(\bullet)$ with $r-1$. As $\operatorname{Tor}_r^S(K,D) = 0$ the connecting homomorphism $\operatorname{Tor}_r^S(K,H) \to \operatorname{Tor}_{r-1}^S(K,A)$ in $(\bullet)$ is an isomorphism. So, for $i = r-2$, $(\bullet)$ gives an exact sequence

$$0 \to \operatorname{Tor}_{r-1}^S(K,D) \to \operatorname{Tor}_{r-1}^S(K,H) \to \operatorname{Tor}_{r-2}^S(K,A) \to \operatorname{Tor}_{r-2}^S(K,D)\ .$$

Using (6.11)b) and $(\bullet)'$ we conclude that $v_{r-2} = 2r-2$ and $w_{r-2} = \binom{r-1}{2}$. So, for $i \ge r-2$, the values of u_i, v_i and w_i are indeed as stated in the table.

Moreover $(\bullet\bullet), (\bullet\bullet)', (\bullet\bullet)''$ and $(\bullet\bullet)'''$ now easily give what is stated in the table on the values u_i, v_i and w_i for $1 \le i \le r-3$, where a_i is as in statement b).

Statement a) is a consequence of (6.4). To prove statement c) observe that the available information on the Betti numbers u_i, v_i and w_i shows that the Hilbert series of A has the form

$$F(t,A) \;=\; \frac{1}{(1-t)^{r+1}} \Big(1 + \sum_{i=1}^{r} (-1)^i \big(u_i t^{i+1} + v_i t^{i+2} + \binom{r-1}{i-1} t^{i+3}\big)\Big)$$

On the other hand, as $\mathcal{C}$ is of n.c.t. (2,1) and as $D(A)$ is the homogeneous coordinate ring of a rational normal curve in $\mathbb{P}_K^{r+2}$ we have (cf. also (5.4)B))

$$F(t,A) = F(t, D(A)) - (2t + t^2) = \frac{1}{(1-t)^2}(1 + (r-1)t + 3t^2 - t^4)\ .$$

Comparing both expressions for $F(t,A)$ we obtain

$$1 + \sum_{i=1}^{r+1} (-1)^i (u_i - v_{i-1}) t^{i+1} = (1-t)^{r-1}\big(1 + (r-1)t + 3t^2\big)\ .$$

¿From this we get statement c). ■

We illustrate this result by a few examples.

(6.13) Examples. A) Consider the non-degenerate rational curve $\mathcal{C} \subseteq \mathbb{P}_K^7$ of degree 9 parametrically given by

$$\mathcal{C} : \big(s^9, s^8t, s^5t^4, s^4t^5, s^3t^6, s^2t^7, st^8, t^9\big)\ .$$

Clearly, $\mathcal{C}$ is a non-singular projection of a rational normal curve in $\mathbb{P}_K^{r+2}$ from a line so that we are in case III or case IV (s. (4.3)c)). The Betti numbers of $\mathcal{C}$ are as follows:

i	1	2	3	4	5	6	7
u_i	18	32	60	24	5	0	0
v_i	0	0	0	15	12	3	0
w_i	1	6	15	20	15	6	1

So, we are actually in the case IV. Moreover $\mathcal{C}$ is contained in a surface $X \subseteq \mathbb{P}^7_K$ of minimal degree.

B) More generally, let $r \geq 4$ and let $\mathcal{C} \subseteq \mathbb{P}^r_K$ be the non-degenerate rational curve of degree $r+2$ given parametrically by

$$\mathcal{C} : \left(s^{r+2}, s^{r+1}t, \ldots, s^4t^{r-2}, st^{r+1}, t^{r+2}\right) ,$$

so that $\mathcal{C}$ is obtained as a non-singular projection from a line of the rational normal curve $\mathcal{C}' \subseteq \mathbb{P}^{r+2}_K$ given parametrically by $\left(s^{r+2}, s^{r+1}t, \ldots, st^{r+1}, t^{r+2}\right)$. An easy calculation sho

(6.14) Remark. A) Assume that we are still in the case IV and keep our previous notations. By (6.12) the minimal graded free resolution of I starts with

$$S^{u_2}(-3) \oplus S^{v_2}(-4) \oplus S^{r-1}(-5) \xrightarrow{\varphi} S^{u_1}(-2) \oplus S^{v_1}(-3) \oplus S(-4) \xrightarrow{\pi} I \to 0 \ .$$

Let $J = \pi\big(S^{u_1}(-2) \oplus S^{v_1}(-3)\big)$ and let $Q \in S_4$ be such that $QS = \pi(S(-4))$. The natural projection of $\varphi(S^{r-1}(-5))$ to $S(-4)$ has the form $L \cdot S(-4)$, where $L \subseteq S$ is an ideal generated by $r-1$ linear forms. It follows $I = (J, Q)$, $L = (J :_S Q)$. As $\operatorname{height}(L) \leq r-1$ it follows $Q \notin L$. From this we get $\operatorname{height}(L) = r-1$, as otherwise $J \subseteq L$ would again imply that J is prime (c.(6.8)) and hence lead to the contradiction that $J = L$. Altogether this implies that L defines a quadrisecant line to $\mathcal{C}$ in $\mathbb{P}^r_K$, the line we already met in the proof of (4.3)c).

B) Keep the notation of part A). We claim that the graded ring S/J is a two-dimensional Cohen-Macaulay ring. First it follows that $I/J \cong S/L(-4)$ because $J :_S Q = L$. So there exists a graded short exact sequence

$$0 \longrightarrow S/L(-4) \longrightarrow S/J \longrightarrow S/I \longrightarrow 0.$$

Because S/L is the coordinate ring of a line in $\mathbb{P}^r_K$ it follows that S/J is a two-dimensional ring such that $N := H^1(S/J)$ is a submodule of the Hartshorne-Rao module H of $\mathcal{C}$. In order to verify our claim we have to show that $N = 0$. Assume the contrary. Then in view of the structure of H (cf.(3.10)) it is to see that N is isomorphic to one of the following modules

$$K(-2),\ \operatorname{Hom}_K\big(S/(\mathbf{x}_0^2, \mathbf{x}_1, \cdots, \mathbf{x}_r)S, K\big)(-2) \ \text{ or } \ H.$$

To this end recall that any non-trivial submodule of H corresponds to a non-trivial epimorphic image of $S/((\mathbf{x}_0, x_1)^2, \mathbf{x}_2, \cdots, \mathbf{x}_r)S$. In each of the three cases there is an isomorphism $\operatorname{Tor}^S_{r+1}(K, N) \cong K(-r-3)$. This is true in the first and last case by (6.11). In the second case this is confirmed by a slight modification of the proof of (6.11) b). As an application of our previous arguments – modified slightly – there is an isomorphism $\operatorname{Tor}^S_r(K, S/J) \cong K(-r-3)$. Let $K_\bullet(-4)$ and $F_\bullet$ denote the Koszul resolution of $S/L(-4)$ and the minimal free resolution of S/J. Then the natural homomorphism $S/L(-4) \longrightarrow S/J$ lifts to a homomorphism $K_\bullet(-4) \xrightarrow{f_\bullet} F_\bullet$ of resolutions. So (see [Ei, Exercise A3.30]) the mapping cone $M := M(f_\bullet)$ provides a free resolution of S/I. Because of $M_r \cong K_{r-1}(-4) \oplus F_r$ and by view of (6.12) it follows that the mapping cone does not provide a minimal free resolution. That means that the homomorphism f_{r-1} splits. But now $F_r \cong K_{r-1}(-4) \cong S(-r-3)$ and therefore the homomorphism $F_r \longrightarrow F_{r-1}$ splits also. But this contradicts the minimality of $F_\bullet$ and therefore $N = 0$. Hence S/J is a Cohen-Macaulay ring, as required.

C) Let $\mathcal{C}$ in $\mathbb{P}^r_K$ denote a reduced irreducible curve of degree d, not contained in any hyperplane. In their paper (see [Gr-L-P]) Gruson, Lazarsfeld and Peskine showed that $\operatorname{reg}(A) \leq d+1-r$ for the coordinate ring A of $\mathcal{C}$. In the case of $d > r+1$ it fails to be $(d+1-r)$-regular if and only if $\mathcal{C}$ is rational and has a $(d+2-r)$-secant line. In the case of our investigations $d = r+2$ and this provides the existence of the 4-secant we have found with different methods. Moreover, as shown in (6.12) we have been able to compute a number of invariants of the free resolution in the particular case $d = r+2$. •

BIBLIOGRAPHY

[B] **Ballico E.**: *On singular curves in positive characteristic;* Math. Nachrichten 141 (1989) 267-273

[Ba-St] **Bayer E., Stillman M.**: *Macaulay, a system for computing in algebraic geometry and commutative algebra*

[Be] **Beauville A.**: *Complex algebraic surfaces*; LMS Lecutre Note 68, Cambridge University Press 1983

[Br_1] **Brodmann M.**: *Bounds on the cohomological Hilbert functions of a projective variety;* Journal of Algebra 109 (1987) 352-380

[Br_2] **Brodmann M.**: *Cohomology of certain projective surfaces with low sectional genus and degree*; in: Commutative Algebra. Algebraic Geometry and Computational Methods (D. Eisenbud, ed.) 173-200; Springer 1999.

[Br-Sh] **Brodmann M., Sharp R.Y.**: *Local cohomology – and algebraic introduction with geometric applications*; Cambridge studies in advanced mathematics No. 60, Cambridge University Press 1998

[Br-V] **Brodmann M., Vogel W.**: *Bounds for the cohomology and the Calstelnuovo Regularity of certain surfaces;* Nagoya Math. Journal 131 (1993) 109-126

[Bru-He] **Bruns W., Herzog J.**: *Cohen-Macaulay rings*; Cambridge studies in advanced mathematics No. 39, Cambridge University Press 1993

[DP] **Del Pezzo P.**: *Sulle superficie dell* n^{mo} *ordine immerse nello spazio a* n *dimensioni*; Rendiconti di Palermo 1887 t. I

[Ei] **Eisenbud D.**: *Commutative Algebra with a View Toward Algebraic Geometry*; Graduate Texts in Mathematics No. 150, Springer, 1994.

[E] **Ein L.**: *Nondenerate surfaces of degree* $n+3$ *in* $\mathbb{P}^n_{\mathbb{C}}$; Journal für die reine und angewandte Mathematik 351 (1984) 1-11

[Ei-Ha] **Eisenbud D., Harris J.**: *On varieties of minimal degree*; (A centennial account); Proc.Sympos. Pure Mathematics 46 (1987) 3-13

[G] **Green M.**: *Koszul homology and the geometry of projective varieties*; Journal of Differential Geometry 19 (1984) 125-171

[G-L] **Green M., Lazarsfeld R.**: *Some results on the syzygus of finite sets and algebraic curves*; Compositio Math. 87 (1988) 301-314

[Gr-L-P] **Gruson L., Lazarsfeld R., Peskine C.**: *On a theorem of Castelnuovo and the equations defining space curves*; Invent. Math. 72 (1983) 491-506

[Ha_1] **Harris J.**: *The genus of space curves;* Math. Annalen 249 (1980) 191-204

[Ha_2] **Harris J.**: *Curves in projective space*; SMS, Séminaire scientifique OTAN, Les presses de l'Université de Montreal, 1982

[Hs] *Hartshorne R.*: *Algebraic geometry;* Springer, New York, 1977

[Ho-St-V] *Hoa T., Stückrad J., Vogel W.*: *Towards a structure theory of projective varieties of degree = codimension +2;* Journal of pure and appl. algebra 71 (1991) 203-231

[N_1] *Nagel U.*: *Über Gradschranken für Syzygien und kohomologische Hilbertfunktionen;* Dissertation, Universität-GHS Paderborn, 1990

[N_2] *Nagel U.*: *On the minimal free resolution of $r+3$ points in projective r-space;* J. Pure Appl. Algebra 96 (1994) 23-38

[S_1] *Schenzel P.*: *Über die freien Auflösungen extremaler Cohen-Macaulay-Ringe*; Journal of Algebra 64 (1980) 93-101

[S_2] *Schenzel P.*: *On Veronesean embeddings and projections of Veronesean varieties;* Archiv der Math. 30 (1978) 301-397

M. Brodmann
Institut für Mathematik
Universität Zürich
Winterthurerstrasse 190
8057 Zürich
Switzerland
email: brodmann@math.unizh.ch

Peter Schenzel
Institut für Informatik
Universität Halle
Kurt-Mothes-Strasse 1
06120 Halle
Deutschland
email: schenzel@informatik.uni-halle.de