Gaußverteilung (Normalverteilung)

Eine Beobachtung:

$$\mathcal{N}(x|\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\frac{(x-\mu)^2}{\sigma^2}}$$

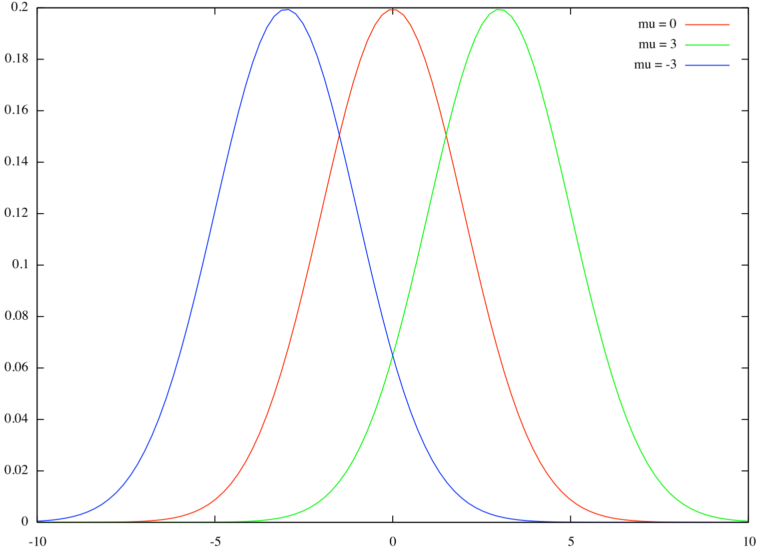
N i.i.d. Beobachtungen:

$$\mathcal{N}(\mathbf{x}|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2}$$

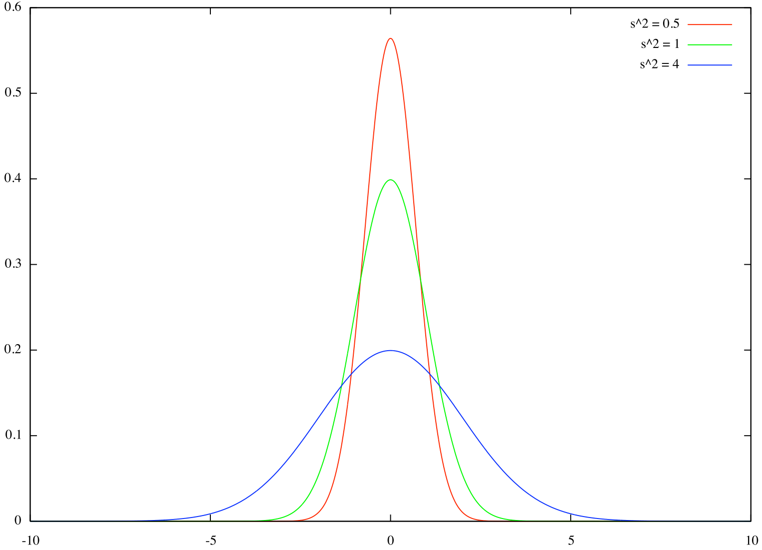
Standardnormalverteilung: $\mu = 0, \sigma^2 = 1$

$$\mathcal{N}(x|0,1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

Einfluß von μ



Einfluß von σ^2



ML Schätzer

$$\hat{\mu}_{ML} = \frac{1}{N}\sum_{i=1}^{N} x_i$$

$$\hat{\sigma^2}_{ML} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu}_{ML})^2$$

$$\hat{\sigma^2}_{ET} = \frac{1}{N-1}\sum_{i=1}^{N}(x_i - \hat{\mu}_{ML})^2$$

Multivariate Gaußverteilung

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \Sigma) = \frac{1}{(2\pi)^{\frac{D}{2}} |\Sigma|^{\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Sigma^{-1}(\mathbf{x}-\boldsymbol{\mu})}$$

$$\mathcal{N}(\mathbf{x}|\boldsymbol{\mu}, \Lambda) = \frac{1}{(2\pi)^{\frac{D}{2}} |\Lambda|^{-\frac{1}{2}}} e^{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \Lambda(\mathbf{x}-\boldsymbol{\mu})}$$

$\boldsymbol{\mu}$: Mittelwertvektor
Σ: Kovarianzmatrix
$\Lambda = \Sigma^{-1}$: Precision-Matrix

Kovarianzmatrix

$$\mathbf{X} = (X_1, X_2, \ldots, X_D)^T$$

$$\Sigma = \begin{pmatrix} \sigma_1^2 & cov(X_1, X_2) & \ldots & cov(X_1, X_D) \\ cov(X_1, X_2) & \sigma_2^2 & \ldots & cov(X_2, X_D) \\ \vdots & \vdots & \vdots & \vdots \\ cov(X_1, X_D) & cov(X_2, X_D) & \ldots & \sigma_D^2 \end{pmatrix}$$

$$cov(X, Y) = E[(X - E[X])(Y - E[Y])]$$

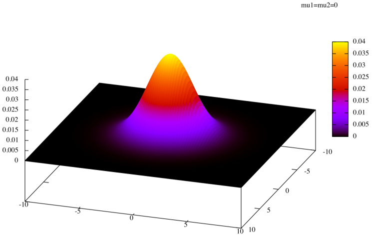
Bivariate Gaußverteilung

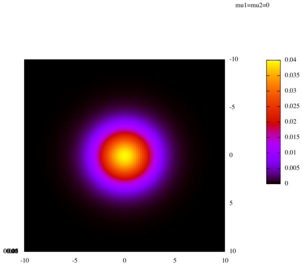
$$\mathcal{N}(x_1, x_2 | \mu_1, \mu_2, C) = \frac{1}{2\pi\sqrt{c_{11}c_{22} - {c_{12}}^2}} e^{(x_1-\mu_1, x_2-\mu_2)\begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}^{-1}\begin{pmatrix} x_1-\mu_1 \\ x_2-\mu_2 \end{pmatrix}}$$

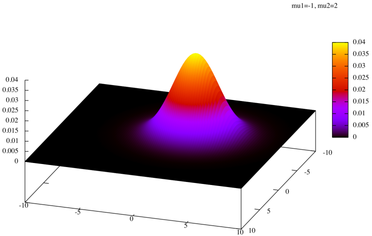
$$\mathcal{N}(x_1, x_2 | \mu_1, \mu_2, \sigma_1^2, \sigma_2^2, \rho) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} e^{\frac{1}{2(1-\rho^2)}\left(\frac{(x_1-\mu_1)^2}{\sigma_1^2} - 2\rho\frac{x_1-\mu_1}{\sigma_1}\frac{x_2-\mu_2}{\sigma_2} + \frac{(x_2-\mu_2)^2}{\sigma_2^2}\right)}$$

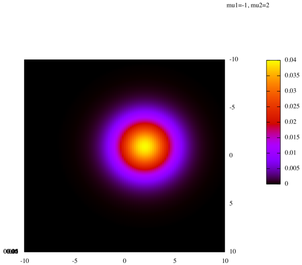
$$C = \Sigma = \begin{pmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{pmatrix}$$

Einfluß von μ

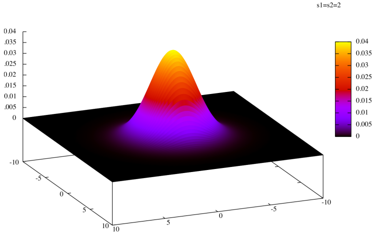


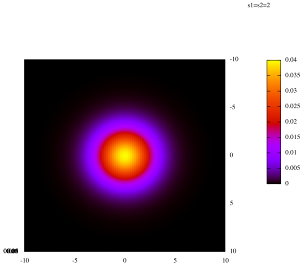


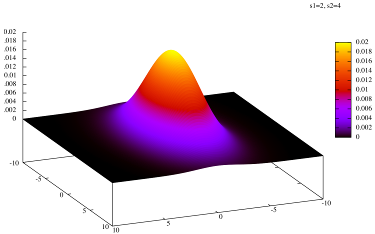


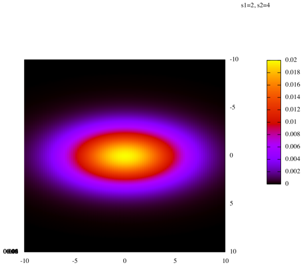


Einfluß von Σ bzw. ρ, $\rho = 0$

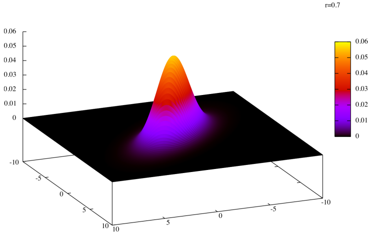


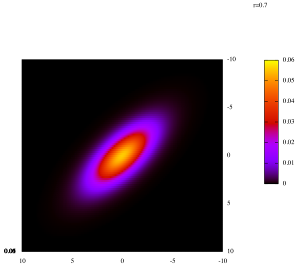


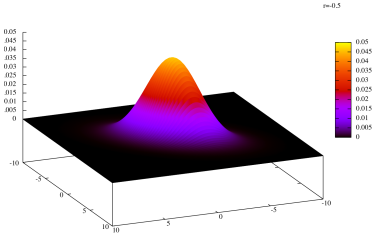


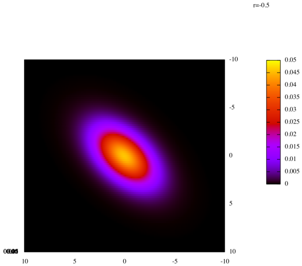


Einfluß von Σ bzw. ρ, $\rho \neq 0$

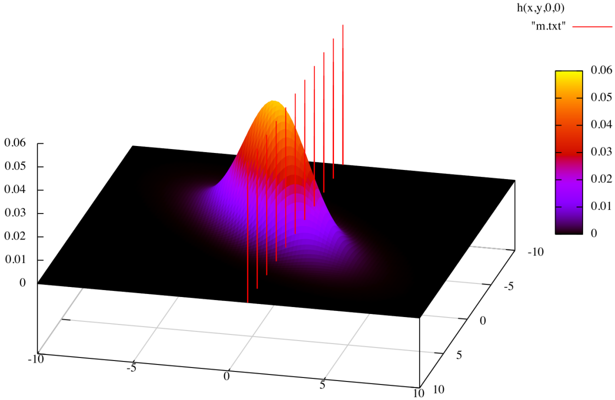




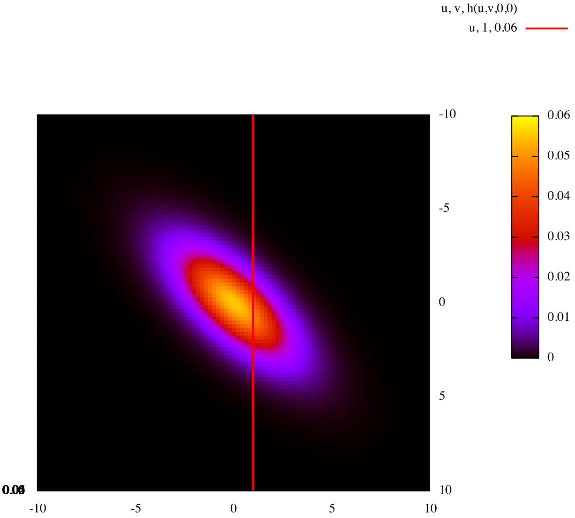




Bedingte Gaußverteilung (1)



Bedingte Gaußverteilung (2)



Bedingte Gaußverteilung (3)

$$\mu_{a|b} = \mu_a + \frac{c_{ab}}{c_{bb}}(x_b - \mu_b)$$

$$\sigma^2_{a|b} = c_{aa} - \frac{c_{ab}c_{ba}}{c_{bb}} = c_{aa}\left(1 - \frac{{c_{ab}}^2}{c_{aa}c_{bb}}\right)$$

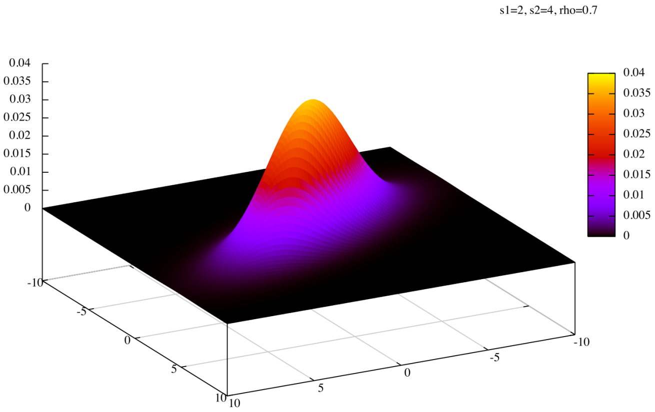
$$\Rightarrow P(x_a|x_b) \sim \mathcal{N}(x_a|\mu_{a|b}, \sigma^2_{a|b})$$

$$\boldsymbol{\mu}_{a|b} = \boldsymbol{\mu}_a + \Sigma_{ab}\Sigma_{bb}^{-1}(\mathbf{x}_b - \boldsymbol{\mu}_b)$$

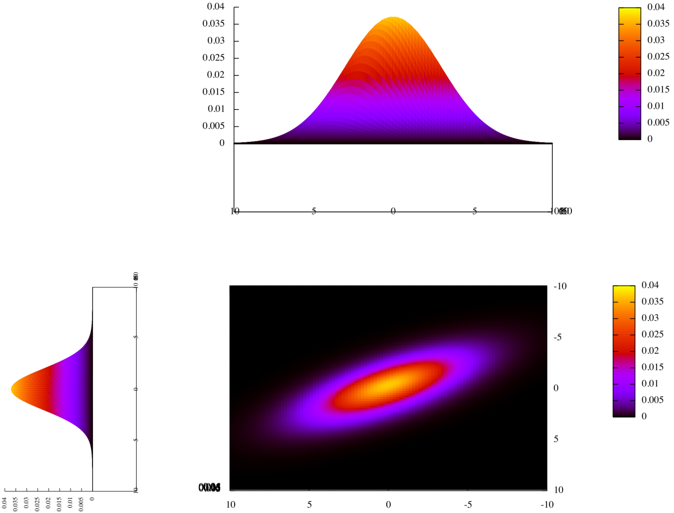
$$\Sigma_{a|b} = \Sigma_{aa} - \Sigma_{ab}\Sigma_{bb}^{-1}\Sigma_{ba}$$

$$\Rightarrow P(\mathbf{x}_a|\mathbf{x}_b) \sim \mathcal{N}(\mathbf{x}_a|\boldsymbol{\mu}_{a|b}, \Sigma_{a|b})$$

Randverteilung einer Gaußverteilung (1)



Randverteilung einer Gaußverteilung (2)



Randverteilung einer Gaußverteilung (3)

$$\begin{aligned} \mu_{a\cdot} &= \mu_a \\ \sigma^2_{a\cdot} &= c_{aa} \end{aligned}$$

$$\Rightarrow P(x_a, \cdot) \sim \mathcal{N}(x_a | \mu_a, c_{aa})$$

$$\begin{aligned} \boldsymbol{\mu}_{a\cdot} &= \boldsymbol{\mu}_a = E[\mathbf{X}_a] \\ \Sigma_{a\cdot} &= \Sigma_{aa} = cov(\mathbf{X}_a) \end{aligned}$$

$$\Rightarrow P(\mathbf{x}_a, \cdot) \sim \mathcal{N}(\mathbf{x}_a | \boldsymbol{\mu}_a, \Sigma_{aa})$$

Bayes'sche Inferenz für Gauß: μ (1)

Likelihood:

$$P(\mathbf{x}|\mu) \sim \mathcal{N}(\mathbf{x}|\mu,\sigma^2) = \frac{1}{(2\pi\sigma^2)^{\frac{N}{2}}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{N}(x_i-\mu)^2}$$

Prior: Gauss

$$P(\mu) \sim \mathcal{N}(\mu|\mu_0,\sigma_0) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2}}$$

$$\Rightarrow P(\mu|\mathbf{x}) = \frac{P(\mathbf{x}|\mu)P(\mu)}{\int_{-\infty}^{\infty} P(\mathbf{x}|\mu)P(\mu)\mathrm{d}\mu}$$

Bayes'sche Inferenz für Gauß μ (2)

Posterior

$$P(\mu|\mathbf{x}) \sim \mathcal{N}(\mu|\mu_{neu}, \sigma^2_{neu})$$

mit

$$\mu_{neu} = \frac{\sigma^2}{N\sigma_0^2 + \sigma^2}\mu_0 + \frac{N\sigma_0^2}{N\sigma_0^2 + \sigma^2}\hat{\mu}_{ML}$$

$$\mu_{neu} = \frac{\sigma^2}{N\sigma_0^2 + \sigma^2}\mu_0 + \frac{N\sigma_0^2}{N\sigma_0^2 + \sigma^2}\left(\frac{1}{N}\sum_{i=1}^{N} x_i\right)$$

$$\frac{1}{\sigma^2_{neu}} = \frac{1}{\sigma_0^2} + \frac{N}{\sigma^2}$$

$$\sigma^2_{neu} = \frac{\sigma_0^2\sigma^2}{\sigma^2 + N\sigma_0^2}$$

Bayes'sche Inferenz für Gauß: σ (1)

Likelihood, $\lambda = \frac{1}{\sigma^2}$:

$$P(\mathbf{x}|\lambda) = \prod_{i=1}^{N} \mathcal{N}(x_i|\mu, \lambda^{-1}) \propto \lambda^{\frac{N}{2}} e^{-\lambda \frac{1}{2} \sum_{i=1}^{N} (x_i - \mu)^2}$$

Prior: Gamma

$$P(\lambda) \sim Gam(\lambda|a_0, b_0) = \frac{b_0^{a_0}}{\Gamma(a_0)} \lambda^{a_0 - 1} e^{-\lambda b_0}$$

$$\Rightarrow P(\lambda|\mathbf{x}) = \frac{P(\mathbf{x}|\lambda)P(\lambda)}{\int_0^\infty P(\mathbf{x}|\lambda)P(\lambda)\mathrm{d}\lambda}$$

Bayes'sche Inferenz für Gauß σ (2)

Posterior:

$$P(\lambda|\mathbf{x}) \sim Gam(\lambda|a_{neu}, b_{neu})$$

mit

$$\begin{aligned}
a_{neu} &= a_0 + \frac{N}{2} \\
b_{neu} &= b_0 + \frac{1}{2}\sum_{i=1}^{N}(x_i - \mu)^2 \\
&= b_0 + \frac{N}{2}\hat{\sigma}^2_{ML}
\end{aligned}$$

Bayes'sche Inferenz für multivariat Gauß

Prior für $\boldsymbol{\mu}$, Λ bekannt: Gauss

$$\mathcal{N}(\boldsymbol{\mu}|\boldsymbol{\mu}_0, \Lambda_0^{-1})$$

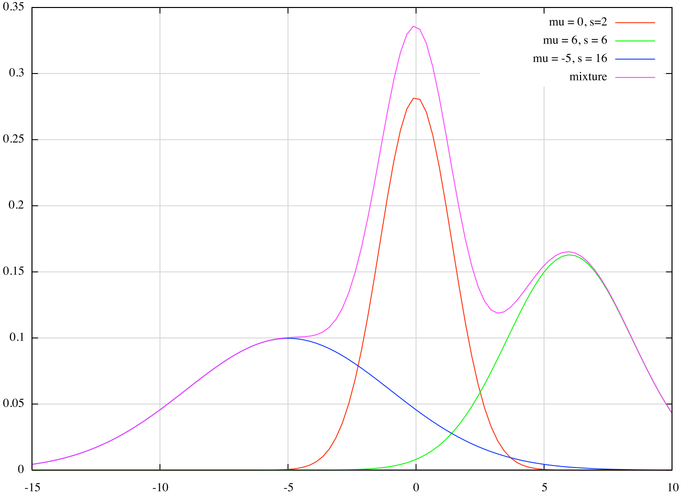
Prior für Λ, $\boldsymbol{\mu}$ bekannt: Wishart

$$\mathcal{W}(\Lambda|W, \nu) = N|\lambda|^{\frac{\nu-D-1}{2}} e^{-\frac{1}{2} Spur(W^{-1}\Lambda)}$$

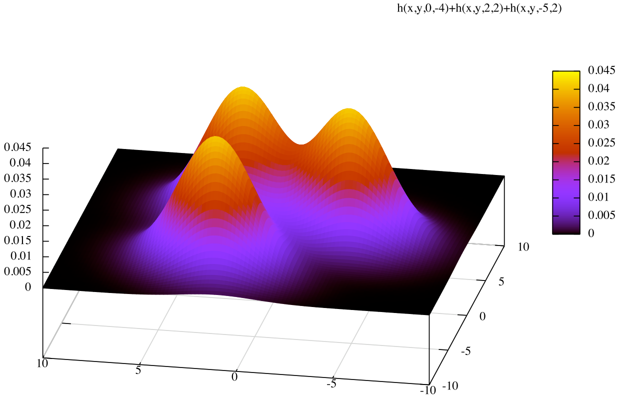
Weder $\boldsymbol{\mu}$ noch Λ bekannt: Normal-Wishart-Verteilung

$$P(\boldsymbol{\mu}, \Lambda|\boldsymbol{\mu}_0, \beta, W, \nu) = \mathcal{N}(\boldsymbol{\mu}|\boldsymbol{\mu}_0, (\beta\Lambda)^{-1})\ \mathcal{W}(\Lambda|W, \nu)$$

Gauß'sche Mischverteilungen (1)



Gauß'sche Mischverteilungen (2)



Gauß'sche Mischverteilungen (3)

Likelihood:

$$P(\mathbf{x}|\{\boldsymbol{\mu}_k\}, \{\Sigma_k\}, \{\pi_k\}) = \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \Sigma_k)$$

$$\begin{aligned} \pi_k &\geq 0 \\ \sum_{k=1}^{K} \pi_k &= 1 \end{aligned}$$

N i.i.d. Beobachtungen:

$$P(\mathbf{x}|\{\boldsymbol{\mu}_k\}, \{\Sigma_k\}, \{\pi_k\}) = \prod_{i=1}^{N} \sum_{k=1}^{K} \pi_k \mathcal{N}(\mathbf{x}_i|\boldsymbol{\mu}_k, \Sigma_k)$$

$\Rightarrow$ Expectation maximization (EM)